

# $C^0$ -ESTIMATES AND SMOOTHNESS OF SOLUTIONS TO THE PARABOLIC EQUATION DEFINED BY KIMURA OPERATORS

CAMELIA A. POP

ABSTRACT. Kimura diffusions serve as a stochastic model for the evolution of gene frequencies in population genetics. Their infinitesimal generator is an elliptic differential operator whose second-order coefficients matrix degenerates on the boundary of the domain. In this article, we consider the inhomogeneous initial-value problem defined by generators of Kimura diffusions, and we establish  $C^0$ -estimates, which allows us to prove that solutions to the inhomogeneous initial-value problem are smooth up to the boundary of the domain where the operator degenerates, even when the initial data is only assumed to be continuous.

## CONTENTS

1. Introduction	1
1.1. Comparison with previous research	4
1.2. Applications of the main results	4
1.3. Outline of the article	5
1.4. Notations and conventions	5
1.5. Acknowledgment	5
2. Anisotropic Hölder spaces	5
2.1. Definition of the anisotropic Hölder spaces	5
2.2. Interpolation inequalities for anisotropic Hölder spaces	8
3. Local a priori Schauder estimates	18
4. Existence and uniqueness of solutions	22
References	27

## 1. INTRODUCTION

The evolution of gene frequencies is one of the central themes of research in population genetics, and one of the natural ways to model the changes of gene frequencies in a population is through the use of Markov chains and their continuous limits. This line of research was initiated by R. Fisher (1922), J. Haldane (1932), S. Wright (1931), and later extended by M. Kimura (1957). The stochastic processes involved in these works are continuous limits of discrete Markov processes, which are solutions to stochastic differential equations whose infinitesimal generator is a degenerate-elliptic partial differential operator. A rigorous understanding of the regularity of solutions to parabolic equations defined by such operators plays a central role in the study of various probabilistic properties of the associated stochastic models.

---

*Date:* June 4, 2014 0:22.

*2010 Mathematics Subject Classification.* Primary 35J70; secondary 60J60.

*Key words and phrases.* Degenerate elliptic operators, anisotropic Hölder spaces, Kimura diffusions, degenerate diffusions.

A wide extension of the generator of continuous limits of the Wright-Fisher model [11, 23, 12, 15, 16, 22, 9, 14] was introduced in the work of C. Epstein and R. Mazzeo [6, 7], where the authors build a suitable Schauder theory to prove existence, uniqueness and optimal regularity of solutions to the inhomogeneous initial-value problem defined by generalized Kimura diffusion operators acting on functions defined on compact manifolds with corners. In our work, we extend the regularity results obtained in [6, 7] by proving a priori local Schauder estimates of solutions, in which we control the higher-order Hölder norm of solutions in terms of their supremum norm (Theorem 1.1). This result allows us to prove in Theorem 1.5 that the solutions are smooth up to the portion of the boundary where the operator degenerates, even when the initial data is only assumed to be continuous, as opposed to Hölder continuous in [6, 7]. In the sequel, we describe our main results and their applications in more detail.

Let  $\mathbb{R}_+ := (0, \infty)$  and  $S_{n,m} := \mathbb{R}_+^n \times \mathbb{R}^m$ , where  $n$  and  $m$  are nonnegative integers such that  $n + m \geq 1$ . While generalized Kimura diffusion operators act on functions defined on compact manifolds with corners [7, §2], from an analytical point of view and due to the fact that we are interested in the local properties of solutions, in our article, we consider a second-order elliptic differential operator of the form

$$\begin{aligned} Lu = & \sum_{i=1}^n (x_i a_{ii}(z) u_{x_i x_i} + b_i(z) u_{x_i}) + \sum_{i,j=1}^n x_i x_j \tilde{a}_{ij}(z) u_{x_i x_j} \\ & + \sum_{i=1}^n \sum_{l=1}^m x_i c_{il}(z) u_{x_i y_l} + \sum_{k,l=1}^m d_{kl}(z) u_{y_k y_l} + \sum_{l=1}^m e_l(z) u_{y_l}, \end{aligned} \quad (1.1)$$

defined for all  $z = (x, y) \in S_{n,m}$  and  $u \in C^2(S_{n,m})$ . Even though the operator  $L$  is defined on  $S_{n,m}$ , we still call  $L$  a generalized Kimura diffusion operator since it preserves the local properties of the Kimura diffusion operators arising in population genetics. The operator  $L$  is not strictly elliptic as we approach the boundary of the domain  $S_{n,m}$ , because the smallest eigenvalue of the second-order coefficient matrix tends to 0 proportional to the distance to the boundary of the domain. For this reason, the sign of the coefficient functions  $b_i(z)$  along  $\partial S_{n,m}$  plays a crucial role in the regularity of solutions, and we always assume that the drift coefficients  $b_i(z)$  are nonnegative functions along  $\partial S_{n,m}$ . The precise technical conditions satisfied by the coefficients of the operator  $L$  are described in Assumption 3.1.

We prove local a priori Schauder estimates of solutions to the inhomogeneous initial-value problem,

$$\begin{aligned} u_t - Lu &= g \quad \text{on } (0, \infty) \times S_{n,m}, \\ u(0, \cdot) &= f \quad \text{on } S_{n,m}, \end{aligned} \quad (1.2)$$

which we then use to prove the smoothness of solutions on  $(0, \infty) \times \bar{S}_{n,m}$ , when the initial data,  $f$ , is assumed to be only *continuous* on  $\bar{S}_{n,m}$ . The inhomogeneous initial-value problem (1.2) on compact manifolds with corners was studied by C. Epstein and R. Mazzeo in [6, 7], where they build suitable anisotropic Hölder spaces [7, Chapter 5] to account for the degeneracy of the operator, and establish existence, uniqueness and optimal regularity of solutions [7, Chapters 3 and 10], under the assumptions that the initial data,  $f$ , and the source function,  $g$ , belong to suitable Hölder spaces, and the coefficients of the differential operator  $L$  are smooth and bounded functions. In our work, we prove that the solutions of the inhomogeneous initial-value problem (1.2) are smooth functions on  $(0, \infty) \times \bar{S}_{n,m}$  and continuous on  $[0, \infty) \times \bar{S}_{n,m}$ , when the coefficients of the operator  $L$  and the source function,  $g$ , are assumed smooth, but the initial data is only assumed to be *continuous*, as opposed to Hölder continuous in [7, §11.2]. In addition, we relax

the assumption in [7] that the coefficients of the operator  $L$  are smooth, and we only require that they are Hölder continuous. Under the new hypotheses, we derive higher-order a priori local Schauder estimates in Theorems 1.1 and 1.2, and we prove existence of solutions in Hölder spaces in Theorem 1.4. The technical definition of the anisotropic Hölder spaces adapted to our framework is given in §2.1.

Our main results are

**Theorem 1.1** (Local a priori Schauder estimates I). *Let  $\alpha \in (0, 1)$  and  $k \in \mathbb{N}$ . Then there is a positive constant,  $r_0 = r_0(\alpha, k, m, n)$ , such that the following hold. Let  $r \in (0, r_0)$  and  $0 < T_0 < T$ . Suppose that the coefficients of the differential operator  $L$  defined in (1.1) satisfy Assumption 3.1. Then, there is a positive constant,  $C = C(\alpha, \delta, k, K, m, n, r, T_0, T)$ , such that for all  $z^0 \in \bar{S}_{n,m}$ , and all functions,  $u \in C_{WF}^{k,2+\alpha}([T_0/2, T] \times \bar{B}_{2r}(z^0))$ , we have that*

$$\|u\|_{C_{WF}^{k,2+\alpha}([T_0, T] \times \bar{B}_r(z^0))} \leq C \left( \|u_t - Lu\|_{C_{WF}^{k,\alpha}([T_0/2, T] \times \bar{B}_{2r}(z^0))} + \|u\|_{C([T_0/2, T] \times \bar{B}_{2r}(z^0))} \right). \quad (1.3)$$

**Theorem 1.2** (Local a priori Schauder estimates II). *Let  $\alpha \in (0, 1)$  and  $k \in \mathbb{N}$ . Then there is a positive constant,  $r_0 = r_0(\alpha, k, m, n)$ , such that the following hold. Let  $r \in (0, r_0)$  and  $T > 0$ . Suppose that the coefficients of the differential operator  $L$  defined in (1.1) satisfy Assumption 3.1. Then, there is a positive constant,  $C = C(\alpha, \delta, k, K, m, n, r, T)$ , such that for all  $z^0 \in \bar{S}_{n,m}$ , and all functions,  $u \in C_{WF}^{k,2+\alpha}([0, T] \times \bar{B}_{2r}(z^0))$ , we have that*

$$\begin{aligned} \|u\|_{C_{WF}^{k,2+\alpha}([0, T] \times \bar{B}_r(z^0))} &\leq C \left( \|u_t - Lu\|_{C_{WF}^{k,\alpha}([0, T] \times \bar{B}_{2r}(z^0))} \right. \\ &\quad \left. + \|u(0, \cdot)\|_{C_{WF}^{k,2+\alpha}(\bar{B}_{2r}(z^0))} + \|u\|_{C([0, T] \times \bar{B}_{2r}(z^0))} \right). \end{aligned} \quad (1.4)$$

**Remark 1.3** (Comparison between Theorems 1.1 and 1.2). Notice that the a priori Schauder estimate (1.3) shows that the  $C_{WF}^{k,2+\alpha}([T_0, T] \times \bar{B}_r(z^0))$ -Hölder norm of the function  $u$  can be controlled in terms of its supremum norm on  $[0, T] \times \bar{B}_{2r}(z^0)$ , while estimate (1.4) also involves the  $C_{WF}^{k,2+\alpha}(\bar{B}_{2r}(z^0))$ -Hölder norm of the initial condition,  $u(0, \cdot)$ . Thus, estimate (1.3) implies that to prove higher-order Hölder regularity of solutions on  $(0, T] \times \bar{S}_{n,m}$ , it is sufficient to establish a control on the supremum norm of the solution  $u$  on  $[0, T] \times \bar{S}_{n,m}$ , a key fact that we use in our proof of the smoothness of solutions to the initial-value problem (1.2) with continuous initial data, in Theorem 1.5.

We now state our results on existence and uniqueness of solutions with Hölder continuous initial data, and with only continuous initial data.

**Theorem 1.4** (Existence and uniqueness of solutions with Hölder continuous initial data). *Let  $\alpha \in (0, 1)$ ,  $k \in \mathbb{N}$  and  $T > 0$ . Suppose that the coefficients of the differential operator  $L$  satisfy Assumption 3.1. Then, there is a positive constant,  $C = C(\alpha, \delta, k, K, m, n, T)$ , such that the following hold. Let  $g \in C_{WF}^{k,\alpha}([0, T] \times \bar{S}_{n,m})$  and  $f \in C_{WF}^{k,2+\alpha}(\bar{S}_{n,m})$ . Then there is a unique solution,  $u \in C_{WF}^{k,2+\alpha}([0, T] \times \bar{S}_{n,m})$ , to the inhomogeneous initial-value problem (1.2), and the function  $u$  satisfies the Schauder estimate,*

$$\|u\|_{C_{WF}^{k,2+\alpha}([0, T] \times \bar{S}_{n,m})} \leq C \left( \|g\|_{C_{WF}^{k,\alpha}([0, T] \times \bar{S}_{n,m})} + \|f\|_{C_{WF}^{k,2+\alpha}(\bar{S}_{n,m})} \right). \quad (1.5)$$

**Theorem 1.5** (Existence and uniqueness of solutions with continuous initial data). *Let  $T > 0$ . Suppose that the coefficients of the differential operator  $L$  satisfy Assumption 3.1, for all  $k \in \mathbb{N}$ . Let  $g \in C^\infty([0, T] \times \bar{S}_{n,m})$  and  $f \in C(\bar{S}_{n,m})$ . Then there is a unique solution,  $u \in C([0, T] \times \bar{S}_{n,m}) \cap C^\infty((0, T] \times \bar{S}_{n,m})$ , to the inhomogeneous initial-value problem (1.2). Moreover, for all*

$\alpha \in (0, 1)$ ,  $k \in \mathbb{N}$  and  $T_0 \in (0, T)$ , there is a positive constant,  $C = C(\alpha, \delta, k, K, m, n, T_0, T)$ , such that

$$\|u\|_{C_{WF}^{k, 2+\alpha}([T_0, T] \times \bar{S}_{n, m})} \leq C \left( \|g\|_{C_{WF}^{k, \alpha}([0, T] \times \bar{S}_{n, m})} + \|f\|_{C(\bar{S}_{n, m})} \right). \quad (1.6)$$

**Remark 1.6** (Coefficients of the operator  $L$ ). The coefficients of the operator  $L$  are assumed to be functions only of the spatial variables, but it is straightforward to extend our results to time-dependent coefficients. Moreover the coefficients are assumed to be bounded functions. This restriction can be removed and replaced by a linear growth of the coefficients in the spatial variables, by incorporating a weight in the definition of the anisotropic Hölder spaces to take into account the growth of the coefficients, similarly to [10, §2].

The proofs of Theorems 1.1 and 1.2 are based on a localization procedure described by N. V. Krylov in the proof of [17, Theorem 8.11.1]. For this method to work, we need interpolation inequalities for our anisotropic Hölder spaces, which we establish in §2.2, and we need global a priori Schauder estimates for model operators, which were established by C. Epstein and R. Mazzeo in [7, Theorem 10.0.2]. These ideas are applicable to a more general functional analytical framework, where global a priori estimates and interpolation inequalities hold, and it was previously employed by P. Feehan and the author in the study of the regularity of solutions defined by a different class of degenerate elliptic equations with applications in Mathematical Finance [10].

**1.1. Comparison with previous research.** C. Epstein and R. Mazzeo prove in [5, Corollary 3.2] smoothness of solutions to the homogeneous initial-value problem defined by the operator  $L$  with continuous, compactly supported data, under the assumption that the operator  $L$  has a special diagonal structure, that is, the coefficients of the cross-terms in (1.1) are 0, and the drift coefficients  $b_i(z)$  are bounded from below by a positive constant on  $\bar{S}_{n, m}$ . In Theorem 1.5 we extend this result by not requiring any special structure of the operator  $L$ , other than the one implied by (1.1) and by Assumption 3.1, and we prove local a priori Schauder estimates in Theorems 1.1 and 1.2.

The results of [5] are further extended in [4, Theorem 1.1], where the authors prove smoothness of solutions to the homogenous Kimura initial-value problem on compact manifolds with corners,  $P$ , when the initial data is assumed to belong to the weighted Sobolev space,  $L^2(P, d\mu_L)$  (for the definition of the weight function  $d\mu_L$  see [4, §2]). The method of the proof of [4] is based on writing the Kimura operator in divergence form and applying the method of Moser iterations [18, 19, 20]. For this method to work, the authors prove that the weight function  $d\mu_L$  is a doubling measure ([4, Proposition 3.1]), and that a suitable  $L^2$ -invariant Poincaré inequality holds ([4, Theorem 3.1]). As a consequence, it is established in [4, Corollaries 4.1 and 4.2] that there is a Hölder exponent,  $\alpha_0 \in (0, 1)$ , such that the  $C_{WF}^{\alpha_0}$ -norm of the solution can be controlled in terms of its sup-norm. Comparing this result with our Theorem 1.1, we prove that for *all*  $\alpha \in (0, 1)$  and for all positive integers,  $k$ , the  $C_{WF}^{k, 2+\alpha}$ -norm of the solution can be controlled in terms of the sup-norm of the initial data. In addition, our method of the proof appears to be more direct and it uses interpolation inequalities adapted to the anisotropic Hölder spaces (§2.2) and a localization procedure due to N. V. Krylov ([17]).

**1.2. Applications of the main results.** We use the existence result in Theorem 1.4 to establish the uniqueness in law and the strong Markov property of solutions to the standard Kimura stochastic differential equation and its singular drift perturbations in [21, §2.3 and §3.2]. The fact that we only require the coefficients of the operator  $L$  to be Hölder continuous, allows us to also assume that the coefficients of the Kimura stochastic differential equation are only Hölder continuous, and so, our results in [21, §2.3 and 3.2] generalize the classical existence and

uniqueness theorems of solutions to stochastic differential equations with Lipschitz continuous coefficients [13, §5.2.B]. They also generalize the existence and uniqueness of weak solutions to a closely related degenerate stochastic differential equations studied in [1, 2]. Moreover the existence, uniqueness and the strong Markov property of weak solutions to Kimura stochastic differential equations and its singular drift perturbation are crucial ingredients in our proof of the Harnack inequality for nonnegative solutions to the homogeneous parabolic equation  $u_t - Lu = 0$ , which we establish in forthcoming work joint with C. Epstein [8, Theorem 7.6].

**1.3. Outline of the article.** In §2, building on the work of C. Epstein and R. Mazzeo [7], we introduce anisotropic Hölder spaces adapted to our framework, and we prove interpolation inequalities for the new Hölder spaces in Proposition 2.1 and Corollary 2.4. In §3 we begin by stating in Assumption 3.1 the conditions satisfied by the coefficients of the operator  $L$ , and we then give the proofs of Theorems 1.1 and 1.2. We prove Theorems 1.4 and 1.5 in §4. In §1.4, we list the notations used in our article.

**1.4. Notations and conventions.** Let  $\mathbb{N} := \{0, 1, 2, 3, \dots\}$ . Given a positive integer  $k$ , we let  $\mathbb{N}^k$  denote the set of multi-indices  $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$ , and we let  $|\alpha| := \alpha_1 + \dots + \alpha_k$ . Given a finite set of elements,  $F$ , we let  $|F|$  denote the cardinal of  $F$ . Let  $B_r(z)$  denote the Euclidean ball centered at a point  $z \in \bar{S}_{n,m}$  of radius  $r$ , relative to the domain  $S_{n,m}$ .

**1.5. Acknowledgment.** The author is indebted to Charles Epstein for suggesting this problem and for many very helpful discussions on this subject.

## 2. ANISOTROPIC HÖLDER SPACES

In this section, we introduce the anisotropic Hölder spaces suitable for obtaining a priori Schauder estimates of solutions to the inhomogeneous initial-value problem (1.2). The Hölder spaces defined in §2.1 are a slight modification of the Hölder spaces introduced by C. Epstein and R. Mazzeo in their study of the existence, uniqueness and regularity of solutions to the parabolic problem defined by generalized Kimura operators [6, 7]. We then establish in §2.2 the interpolation inequalities satisfied by the anisotropic Hölder spaces. These properties will be a main ingredient in the proofs of the results in our article.

**2.1. Definition of the anisotropic Hölder spaces.** Following [7, Chapter 5], we need to first introduce a *distance function*,  $\rho$ , which takes into account the degeneracy of the second-order coefficient matrix of the operator  $L$ . We let

$$\rho((t^0, z^0), (t, z)) := \rho_0(z^0, z) + \sqrt{|t^0 - t|}, \quad \forall (t^0, z^0), (t, z) \in [0, \infty) \times \bar{S}_{n,m}, \quad (2.1)$$

where  $\rho_0$  is a distance function in the spatial variables. Because our domain  $S_{n,m}$  is unbounded, as opposed to the compact manifolds considered in [7], the properties of the distance function  $\rho_0(z^0, z)$  depend on whether the points  $z^0$  and  $z$  are in a neighborhood of the boundary of  $S_{n,m}$ , or far away from the boundary of  $S_{n,m}$ . For any set of indices,  $I \subseteq \{1, \dots, n\}$ , we let

$$M_I := \{z = (x, y) \in S_{n,m} : x_i \in (0, 1) \text{ for all } i \in I, \text{ and } x_j \in (1, \infty) \text{ for all } j \in I^c\}, \quad (2.2)$$

where we denote  $I^c := \{1, 2, \dots, n\} \setminus I$ . The distance function  $\rho_0$  has the property that there is a positive constant,  $c = c(n, m)$ , such that for all sets of indices,  $I, J \subseteq \{1, \dots, n\}$ , and all  $z^0 \in \bar{M}_I$

and  $z \in \bar{M}_J$ , we have that

$$\begin{aligned} & c \left( \max_{i \in I \cap J} \left| \sqrt{x_i^0} - \sqrt{x_i} \right| + \max_{j \in (I \cap J)^c} |x_j^0 - x_j| + \max_{l \in \{1, \dots, m\}} |y_l^0 - y_l| \right) \\ & \leq \rho(z^0, z) \\ & \leq c^{-1} \left( \max_{i \in I \cap J} \left| \sqrt{x_i^0} - \sqrt{x_i} \right| + \max_{j \in (I \cap J)^c} |x_j^0 - x_j| + \max_{l \in \{1, \dots, m\}} |y_l^0 - y_l| \right). \end{aligned} \quad (2.3)$$

Let  $k \in \mathbb{N}$ ,  $T > 0$ , and  $U \subseteq S_{n,m}$ . We let  $C^k([0, T] \times U)$  denote the space consisting of functions  $u : [0, T] \times U \rightarrow \mathbb{R}$  that are continuous and locally bounded, and we let  $C^k([0, T] \times \bar{U})$  denote the Banach space of functions  $u : [0, T] \times \bar{U} \rightarrow \mathbb{R}$ , with continuous, bounded derivatives up to order  $k$ , endowed with the norm,

$$\|u\|_{C^k([0, T] \times \bar{U})} := \sum_{\substack{\tau \in \mathbb{N}, \zeta \in \mathbb{N}^{n+m} \\ 2\tau + |\zeta| \leq k}} \sup_{(t, z) \in [0, T] \times \bar{U}} |D_t^\tau D_z^\zeta u(t, z)|.$$

We let  $C^\infty([0, T] \times \bar{U})$  be the space of smooth functions  $u : [0, T] \times \bar{U} \rightarrow \mathbb{R}$ , with continuous and bounded derivatives of all orders, and we let  $C_c^\infty([0, T] \times \bar{U})$  be the space of smooth functions with compact support in  $[0, T] \times \bar{U}$ .

We recall the definition of the standard parabolic Hölder spaces [17, §8.5]. Let  $\alpha \in (0, 1)$ . Then  $C^{0, \alpha}([0, T] \times \bar{U})$  denotes the Hölder spaces of functions  $u : [0, T] \times \bar{U} \rightarrow \mathbb{R}$ , such that

$$\|u\|_{C^{0, \alpha}([0, T] \times \bar{U})} := \|u\|_{C^0([0, T] \times \bar{U})} + \sup_{\substack{(t^0, z^0), (t, z) \in [0, T] \times \bar{U} \\ (t^0, z^0) \neq (t, z)}} \frac{|u(t^0, z^0) - u(t, z)|}{(|z - z^0| + \sqrt{|t - t^0|})^\alpha}.$$

The space  $C^{k, \alpha}([0, T] \times \bar{U})$  consists of functions  $u : [0, T] \times \bar{U} \rightarrow \mathbb{R}$ , such that for all  $\tau \in \mathbb{N}$  and  $\zeta \in \mathbb{N}^{n+m}$  satisfying the property that  $2\tau + |\zeta| \leq k$ , we have

$$D_t^\tau D_z^\zeta u \in C^{0, \alpha}([0, T] \times \bar{U}),$$

and we endow the space  $C^{k, \alpha}([0, T] \times \bar{U})$  with the norm:

$$\|u\|_{C^{k, \alpha}([0, T] \times \bar{U})} := \sum_{\substack{\tau \in \mathbb{N}, \zeta \in \mathbb{N}^{n+m} \\ 2\tau + |\zeta| \leq k}} \|D_t^\tau D_z^\zeta u\|_{C^{0, \alpha}([0, T] \times \bar{U})}.$$

Following [7, §5.2.4], we can now introduce the anisotropic Hölder spaces suitable to establish a priori Schauder estimates for solutions to the inhomogeneous initial-value problem (1.2). We let  $C_{WF}^{0, \alpha}([0, T] \times \bar{U})$  be the Hölder space consisting of continuous functions,  $u : [0, T] \times \bar{U} \rightarrow \mathbb{R}$ , such that the following norm is finite

$$\|u\|_{C_{WF}^{0, \alpha}([0, T] \times \bar{U})} := \|u\|_{C^0([0, T] \times \bar{U})} + \sup_{\substack{(t^0, z^0), (t, z) \in [0, T] \times \bar{U} \\ (t^0, z^0) \neq (t, z)}} \frac{|u(t^0, z^0) - u(t, z)|}{\rho^\alpha((t^0, z^0), (t, z))}.$$

We let  $C_{WF}^{k, \alpha}([0, T] \times \bar{U})$  denote the Hölder space containing functions,  $u \in C^k([0, T] \times \bar{U})$ , such that the derivatives  $D_t^\tau D_z^\zeta$  belong to the space  $C_{WF}^{0, \alpha}([0, T] \times \bar{U})$ , for all  $\tau \in \mathbb{N}$  and  $\zeta \in \mathbb{N}^{n+m}$ , such that  $2\tau + |\zeta| \leq k$ . We endow the space  $C_{WF}^{k, \alpha}([0, T] \times \bar{U})$  with the norm,

$$\|u\|_{C_{WF}^{k, \alpha}([0, T] \times \bar{U})} := \sum_{\substack{\tau \in \mathbb{N}, \zeta \in \mathbb{N}^{n+m} \\ 2\tau + |\zeta| \leq k}} \|D_t^\tau D_z^\zeta u\|_{C_{WF}^{0, \alpha}([0, T] \times \bar{U})}.$$



We fix a set of indices,  $I \subseteq \{1, \dots, n\}$ . Let  $U$  be a set such that  $U \subseteq M_I$ . We let  $C_{WF}^{0,2+\alpha}([0, T] \times \bar{U})$  denote the Hölder space of functions,  $u \in C_{WF}^{1,\alpha}([0, T] \times \bar{U}) \cap C^2([0, T] \times U)$ , such that

$$u_t \in C_{WF}^{0,\alpha}([0, T] \times \bar{U}),$$

and such that the functions,

$$\begin{aligned} \sqrt{x_i x_j} u_{x_i x_j}, \sqrt{x_i} u_{x_i y_l}, u_{y_l y_k} &\in C_{WF}^{0,\alpha}([0, T] \times \bar{U}), \quad \forall i, j \in I, \quad \forall l, k = 1, \dots, m, \\ \sqrt{x_i} u_{x_i x_j}, u_{x_j x_k} &\in C_{WF}^{0,\alpha}([0, T] \times \bar{U}), \quad \forall i \in I, \quad \forall j, k \in I^c. \end{aligned}$$

We endowed the space  $C_{WF}^{0,2+\alpha}([0, T] \times \bar{U})$  with the norm,

$$\begin{aligned} \|u\|_{C_{WF}^{0,2+\alpha}([0, T] \times \bar{U})} &:= \|u\|_{C_{WF}^{1,\alpha}([0, T] \times \bar{U})} + \sum_{i,j \in I} \|\sqrt{x_i x_j} u_{x_i x_j}\|_{C_{WF}^{0,\alpha}([0, T] \times \bar{U})} \\ &+ \sum_{l,k=1}^m \|u_{y_l y_k}\|_{C_{WF}^{0,\alpha}([0, T] \times \bar{U})} + \sum_{i \in I} \sum_{j \in I^c} \|\sqrt{x_i} u_{x_i x_j}\|_{C_{WF}^{0,\alpha}([0, T] \times \bar{U})} \\ &+ \sum_{i \in I} \sum_{l=1}^m \|\sqrt{x_i} u_{x_i y_l}\|_{C_{WF}^{0,\alpha}([0, T] \times \bar{U})} + \sum_{i,j \in I^c} \|u_{x_i x_j}\|_{C_{WF}^{0,\alpha}([0, T] \times \bar{U})} \\ &+ \sum_{i \in I^c} \sum_{l=1}^m \|u_{x_i y_l}\|_{C_{WF}^{0,\alpha}([0, T] \times \bar{U})} + \|u_t\|_{C_{WF}^{0,\alpha}([0, T] \times \bar{U})}. \end{aligned}$$

We now consider the case when  $U$  is an arbitrary set in  $S_{n,m}$ . We let  $C_{WF}^{0,2+\alpha}([0, T] \times \bar{U})$  denote the Hölder space consisting of functions  $u \in C^2([0, T] \times U)$ , satisfying the property that

$$u \upharpoonright_{\bar{U} \cap \bar{M}_I} \in C_{WF}^{0,2+\alpha}([0, T] \times (\bar{U} \cap \bar{M}_I)), \quad \forall I \subseteq \{1, \dots, n\}.$$

We endow the Hölder space  $C_{WF}^{0,2+\alpha}([0, T] \times \bar{U})$  with the norm

$$\|u\|_{C_{WF}^{0,2+\alpha}([0, T] \times \bar{U})} = \sum_{I \subseteq \{1, \dots, n\}} \|u\|_{C_{WF}^{0,2+\alpha}([0, T] \times (\bar{U} \cap \bar{M}_I))}.$$

We let  $C_{WF}^{k,2+\alpha}([0, T] \times \bar{U})$  be the space of functions  $u \in C^k([0, T] \times U)$ , satisfying the property that

$$D_t^\tau D_z^\zeta u \in C_{WF}^{0,2+\alpha}([0, T] \times \bar{U}), \quad \forall \tau \in \mathbb{N}, \forall \zeta \in \mathbb{N}^{n+m} \text{ such that } 2\tau + |\zeta| \leq k,$$

and we endow it with the norm

$$\|u\|_{C_{WF}^{k,2+\alpha}([0, T] \times \bar{U})} := \sum_{\substack{\tau \in \mathbb{N}, \zeta \in \mathbb{N}^{n+m} \\ 2\tau + |\zeta| \leq k}} \|D_t^\tau D_z^\zeta u\|_{C_{WF}^{0,2+\alpha}([0, T] \times \bar{U})}.$$

When  $k = 0$ , we write for brevity  $C([0, T] \times \bar{U})$ ,  $C^\alpha([0, T] \times \bar{U})$ ,  $C_{WF}^\alpha([0, T] \times \bar{U})$  and  $C_{WF}^{2+\alpha}([0, T] \times \bar{U})$ , instead of  $C^0([0, T] \times \bar{U})$ ,  $C^{0,\alpha}([0, T] \times \bar{U})$ ,  $C_{WF}^{0,\alpha}([0, T] \times \bar{U})$  and  $C_{WF}^{0,2+\alpha}([0, T] \times \bar{U})$ .

The elliptic Hölder spaces  $C^{k,\alpha}(\bar{U})$ ,  $C_{WF}^{k,\alpha}(\bar{U})$  and  $C_{WF}^{k,2+\alpha}(\bar{U})$  are defined analogously to their parabolic counterparts, and so, we omit their definitions for brevity.

**2.2. Interpolation inequalities for anisotropic Hölder spaces.** To prove the a priori Schauder estimates in Theorems 1.1 and 1.2, and the existence and uniqueness of solutions in Theorems 1.4 and 1.5, we need to develop suitable interpolation inequalities for the anisotropic Hölder spaces introduced in §2.1.

For any set of indices,  $I \subseteq \{1, \dots, n\}$ , we let

$$M'_I := \{z = (x, y) \in S_{n,m} : x_i \in (0, 1) \text{ for all } i \in I, \text{ and } x_j \in (1/2, \infty) \text{ for all } j \in I^c\}, \quad (2.4)$$

$$M''_I := \{z = (x, y) \in S_{n,m} : x_i \in (0, 2) \text{ for all } i \in I, \text{ and } x_j \in (1/4, \infty) \text{ for all } j \in I^c\}, \quad (2.5)$$

where we recall that  $I^c := \{1, 2, \dots, n\} \setminus I$ . Comparing the sets defined in (2.4) and (2.5) with the set  $M_I$  defined in (2.2), we have that  $M_I \subset M'_I \subset M''_I$ .

We begin with

**Proposition 2.1** (Interpolation inequalities). *Let  $T > 0$  and  $\alpha \in (0, 1)$ . Then there are positive constants,  $C = C(\alpha, m, n, T)$  and  $m_0 = m_0(\alpha, m, n)$ , such that for any function,  $u \in C_{WF}^{2+\alpha}([0, T] \times \bar{S}_{n,m})$ , and for all  $\varepsilon \in (0, 1)$ , the following hold:*

$$\|u\|_{C_{WF}^\alpha([0, T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{2+\alpha}([0, T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_0} \|u\|_{C([0, T] \times \bar{S}_{n,m})}, \quad (2.6)$$

$$\|u_{x_i}\|_{C([0, T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{2+\alpha}([0, T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_0} \|u\|_{C([0, T] \times \bar{S}_{n,m})}, \quad (2.7)$$

$$\|u_{y_l}\|_{C([0, T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{2+\alpha}([0, T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_0} \|u\|_{C([0, T] \times \bar{S}_{n,m})}, \quad (2.8)$$

$$\|u_t\|_{C([0, T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{2+\alpha}([0, T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_0} \|u\|_{C([0, T] \times \bar{S}_{n,m})}. \quad (2.9)$$

Let  $I \subseteq \{1, \dots, n\}$ , and assume in addition that the function  $u$  has support in  $[0, T] \times \bar{M}''_I$ . Then, for all  $l, k = 1, \dots, m$ , the following hold:

$$\|\sqrt{x_i x_j} u_{x_i x_j}\|_{C([0, T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{2+\alpha}([0, T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_0} \|u\|_{C([0, T] \times \bar{S}_{n,m})}, \quad \forall i, j \in I, \quad (2.10)$$

$$\|\sqrt{x_i} u_{x_i x_j}\|_{C([0, T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{2+\alpha}([0, T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_0} \|u\|_{C([0, T] \times \bar{S}_{n,m})}, \quad \forall i \in I, j \in I^c, \quad (2.11)$$

$$\|\sqrt{x_i} u_{x_i y_l}\|_{C([0, T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{2+\alpha}([0, T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_0} \|u\|_{C([0, T] \times \bar{S}_{n,m})}, \quad \forall i \in I, \quad (2.12)$$

$$\|u_{x_i x_j}\|_{C([0, T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{2+\alpha}([0, T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_0} \|u\|_{C([0, T] \times \bar{S}_{n,m})}, \quad \forall i, j \in I^c \quad (2.13)$$

$$\|u_{y_l y_k}\|_{C([0, T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{2+\alpha}([0, T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_0} \|u\|_{C([0, T] \times \bar{S}_{n,m})}, \quad (2.14)$$

and we also have that

$$\|x_i u_{x_i}\|_{C_{WF}^\alpha([0, T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{2+\alpha}([0, T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_0} \|u\|_{C([0, T] \times \bar{S}_{n,m})}, \quad \forall i \in I, \quad (2.15)$$

$$\|u_{x_j}\|_{C_{WF}^\alpha([0, T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{2+\alpha}([0, T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_0} \|u\|_{C([0, T] \times \bar{S}_{n,m})}, \quad \forall j \in I^c, \quad (2.16)$$

$$\|u_{y_l}\|_{C_{WF}^\alpha([0, T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{2+\alpha}([0, T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_0} \|u\|_{C([0, T] \times \bar{S}_{n,m})}. \quad (2.17)$$

We give the technical proof of Proposition 2.1 at the end of the section.

**Remark 2.2** (The hypothesis in Proposition 2.1 about the support of the function  $u$ ). To prove inequalities (2.10)-(2.17), we assume that the function  $u$  has support in  $[0, T] \times \bar{M}''_I$ , for some set of indices,  $I \subseteq \{1, \dots, n\}$ . This is only because of the fact that the weights of the derivatives in the  $x_i$ -coordinates of  $u$  differ depending on whether the index  $i$  belongs to the set  $I$ , or its complement,  $I^c$ , that is, the  $x_i$ -coordinate is small or large, respectively. Inequalities (2.14) and (2.17) hold without the assumption that the support of  $u$  is contained in  $[0, T] \times \bar{M}''_I$ .



**Remark 2.3** (Comparison between the interpolation inequalities in the standard Hölder spaces and in the anisotropic Hölder spaces). Notice that Proposition 2.1 does not establish the analogue of [17, Inequality (8.8.4)], that is,

$$[u_{x_i}]_{C^\alpha([0,T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C^{2,\alpha}([0,T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_0} \|u\|_{C([0,T] \times \bar{S}_{n,m})}.$$

This is replaced by the weighted inequality (2.15), due to the fact that the anisotropic Hölder space  $C_{WF}^{2+\alpha}([0,T] \times \bar{S}_{n,m})$  allows for more general functions than the standard Hölder space  $C^{2,\alpha}([0,T] \times \bar{S}_{n,m})$ .

We have the following corollary to Proposition 2.1, which contains the interpolation inequalities for the higher-order anisotropic Hölder spaces.

**Corollary 2.4** (Higher-order interpolation inequalities). *Let  $\alpha \in (0, 1)$ ,  $k \in \mathbb{N}$  and  $T > 0$ . Then there are positive constants,  $C = C(\alpha, k, m, n, T)$  and  $m_k = m_k(\alpha, k, m, n)$ , such that for any function,  $u \in C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})$ , the following hold. Let  $\tau \in \mathbb{N}$  and  $\zeta \in \mathbb{N}^{n+m}$  be such that  $2\tau + |\zeta| \leq k$ , then for all  $\varepsilon \in (0, 1)$ , we have*

$$\|D_t^\tau D_z^\zeta u\|_{C_{WF}^\alpha([0,T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_k} \|u\|_{C([0,T] \times \bar{S}_{n,m})}, \quad (2.18)$$

$$\|D_t^\tau D_z^\zeta u_{x_i}\|_{C([0,T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_k} \|u\|_{C([0,T] \times \bar{S}_{n,m})}, \quad (2.19)$$

$$\|D_t^\tau D_z^\zeta u_{y_l}\|_{C([0,T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_k} \|u\|_{C([0,T] \times \bar{S}_{n,m})}, \quad (2.20)$$

$$\|D_t^\tau D_z^\zeta u_t\|_{C([0,T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_k} \|u\|_{C([0,T] \times \bar{S}_{n,m})}. \quad (2.21)$$

Let  $I \subseteq \{1, \dots, n\}$ , and assume in addition that the function  $u$  has support in  $[0, T] \times \bar{M}_I''$ . Then, for all  $l, p = 1, \dots, m$ , the following hold

$$\|\sqrt{x_i x_j} D_t^\tau D_z^\zeta u_{x_i x_j}\|_{C([0,T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_k} \|u\|_{C([0,T] \times \bar{S}_{n,m})}, \quad \forall i, j \in I, \quad (2.22)$$

$$\|\sqrt{x_i} D_t^\tau D_z^\zeta u_{x_i x_j}\|_{C([0,T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_k} \|u\|_{C([0,T] \times \bar{S}_{n,m})}, \quad \forall i \in I, j \in I^c, \quad (2.23)$$

$$\|\sqrt{x_i} D_t^\tau D_z^\zeta u_{x_i y_l}\|_{C([0,T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_k} \|u\|_{C([0,T] \times \bar{S}_{n,m})}, \quad \forall i \in I, \quad (2.24)$$

$$\|D_t^\tau D_z^\zeta u_{x_i x_j}\|_{C([0,T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_k} \|u\|_{C([0,T] \times \bar{S}_{n,m})}, \quad \forall i, j \in I^c, \quad (2.25)$$

$$\|D_t^\tau D_z^\zeta u_{y_l y_p}\|_{C([0,T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_k} \|u\|_{C([0,T] \times \bar{S}_{n,m})}, \quad (2.26)$$

and we also have that

$$\|x_i D_t^\tau D_z^\zeta u_{x_i}\|_{C_{WF}^\alpha([0,T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_k} \|u\|_{C([0,T] \times \bar{S}_{n,m})}, \quad \forall i \in I, \quad (2.27)$$

$$\|D_t^\tau D_z^\zeta u_{x_j}\|_{C_{WF}^\alpha([0,T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_k} \|u\|_{C([0,T] \times \bar{S}_{n,m})}, \quad \forall j \in I^c, \quad (2.28)$$

$$\|D_t^\tau D_z^\zeta u_{y_l}\|_{C_{WF}^\alpha([0,T] \times \bar{S}_{n,m})} \leq \varepsilon \|u\|_{C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})} + C\varepsilon^{-m_k} \|u\|_{C([0,T] \times \bar{S}_{n,m})}. \quad (2.29)$$

*Proof.* From the definition of the anisotropic Hölder spaces in §2.1, the function  $D_t^\tau D_z^\zeta u$  belongs to the Hölder space  $C_{WF}^{0,2+\alpha}([0,T] \times \bar{U})$ , for all  $\tau \in \mathbb{N}$ ,  $\zeta \in \mathbb{N}^{n+m}$  with the property that  $2\tau + |\zeta| \leq k$ , and so, Proposition 2.1 applies to the function  $D_t^\tau D_z^\zeta u$ . Thus, it is sufficient to show that

inequality (2.18) holds, because the rest of the interpolation inequalities, (2.19)-(2.29), follow by applying Proposition 2.1 to the function  $D_t^\tau D_z^\zeta u$  instead of  $u$ , and using inequality (2.18). We now prove inequality (2.18).

Applying inequality (2.6) to the function  $D_t^\tau D_z^\zeta u$  instead of  $u$ , and then applying inequalities (2.7), (2.8) and (2.9) to the derivatives of the function  $u$ , we obtain that there are positive constants,  $C = C(\alpha, k, m, n, T)$  and  $m_k = m_k(\alpha, k, m, n)$ , such that for all  $\varepsilon \in (0, 1)$ , inequality (2.6) holds. This completes the proof.  $\square$

We use Corollary 2.4 to prove the following Lemmas 2.5 and 2.7. Both Lemmas 2.5 and 2.7 are technical estimates used in the proofs of Theorems 1.1 and 1.4, respectively.

**Lemma 2.5** (Estimate of  $Lu$ ). *Let  $\alpha \in (0, 1)$ ,  $k \in \mathbb{N}$  and  $T > 0$ . Let  $I \subseteq \{1, \dots, n\}$  and let  $U$  be an open set in  $M_I''$ . Suppose that the coefficients of the operator  $L$  satisfy property (3.2). Then there are positive constants,  $C = C(\alpha, k, K, m, n)$  and  $m_k = m_k(\alpha, k, m, n)$ , such that for all functions  $u \in C_{WF}^{k, 2+\alpha}([0, T] \times \bar{S}_{n, m})$  with support in  $[0, T] \times \bar{U}$ , we have that*

$$\|Lu\|_{C_{WF}^{k, \alpha}([0, T] \times \bar{U})} \leq (\Lambda + C\varepsilon)\|u\|_{C_{WF}^{k, 2+\alpha}([0, T] \times \bar{U})} + C\varepsilon^{-m_k}\|u\|_{C([0, T] \times \bar{U})}. \quad (2.30)$$

where the positive constant  $\Lambda$  is given by

$$\begin{aligned} \Lambda := & \sum_{i \in I} \|a_{ii}\|_{C(\bar{U})} + \sum_{i \in I^c} \|x_i a_{ii}\|_{C(\bar{U})} + \sum_{i, j \in I} \|\tilde{a}_{ij}\|_{C(\bar{U})} \\ & + \sum_{i \in I, j \in I^c} \|x_j \tilde{a}_{ij}\|_{C(\bar{U})} + \sum_{j \in I, i \in I^c} \|x_i \tilde{a}_{ij}\|_{C(\bar{U})} + \sum_{i=1}^n \|b_i\|_{C(\bar{U})} \\ & + \sum_{i \in I} \sum_{l=1}^m \|c_{il}\|_{C(\bar{U})} + \sum_{i \in I^c} \sum_{l=1}^m \|x_i c_{il}\|_{C(\bar{U})} + \sum_{l, p=1}^m \|d_{lp}\|_{C(\bar{U})} + \sum_{l=1}^m \|e_l\|_{C(\bar{U})}. \end{aligned} \quad (2.31)$$

**Remark 2.6.** Given a set of indices  $I \subseteq \{1, \dots, n\}$  and a function  $u$  with support in  $[0, T] \times \bar{M}_I''$ , notice that there is no loss of generality in assuming that  $I = \{1, \dots, n\}$ , because for any other set  $I \neq \{1, \dots, n\}$ , the  $x_j$ -variables, for  $j \in I^c$ , can be treated as the  $y_l$ -variables, for all  $l = 1, \dots, m$ . That is, by relabeling the variables, we may replace the space  $S_{n, m}$  by  $S_{n', m'}$ , where  $n' = |I|$  and  $m' = m + n - |I|$ , so that the support of the function  $u$  becomes a subset of  $\bar{M}_{\{1, \dots, n'\}}''$ . Note that  $n'$  may be zero.

*Proof of Lemma 2.5.* Remark 2.6 shows that we may assume without loss of generality that  $U \subseteq M_{\{1, \dots, n\}}''$ . Under the assumption that  $U \subseteq M_{\{1, \dots, n\}}''$ , the definition of the constant  $\Lambda$  in (2.31) simplifies to

$$\begin{aligned} \Lambda := & \sum_{i=1}^n \|a_{ii}\|_{C(\bar{U})} + \sum_{i, j=1}^n \|\tilde{a}_{ij}\|_{C(\bar{U})} + \sum_{i=1}^n \|b_i\|_{C(\bar{U})} \\ & + \sum_{i=1}^n \sum_{l=1}^m \|c_{il}\|_{C(\bar{U})} + \sum_{l, p=1}^m \|d_{lp}\|_{C(\bar{U})} + \sum_{l=1}^m \|e_l\|_{C(\bar{U})}. \end{aligned} \quad (2.32)$$

Let  $\tau \in \mathbb{N}$  and  $\zeta \in \mathbb{N}^{n+m}$  be such that  $2\tau + |\zeta| \leq k$ . We need to show that the functions  $D_t^\tau D_z^\zeta(Lu)$  have  $C_{WF}^\alpha([0, T] \times \bar{U})$ -Hölder norm bounded by the right-hand side of inequality (2.30). That is, for all  $i, j = 1, \dots, n$  and all  $l, p = 1, \dots, m$ , the functions

$$\begin{aligned} & D_t^\tau D_z^\zeta(x_i a_{ii} u_{x_i x_i}), \quad D_t^\tau D_z^\zeta(x_i x_j \tilde{a}_{ij} u_{x_i x_j}), \quad D_t^\tau D_z^\zeta(b_i u_{x_i}), \\ & D_t^\tau D_z^\zeta(x_i c_{il} u_{x_i y_l}), \quad D_t^\tau D_z^\zeta(d_{lp} u_{y_l y_p}), \quad D_t^\tau D_z^\zeta(e_l u_{y_l}) \end{aligned} \quad (2.33)$$

have  $C_{WF}^\alpha([0, T] \times \bar{U})$ -Hölder norms bounded by the right-hand side of inequality (2.30). We will prove this fact for the function  $D_t^\tau D_z^\zeta(x_i a_{ii} u_{x_i x_i})$ , but the analysis is the same for all the remaining functions in (2.33), and so, we omit the details for brevity. Direct calculations give us

$$\begin{aligned} D_t^\tau D_z^\zeta(x_i a_{ii} u_{x_i x_i}) &= x_i a_{ii}(z) \left( D_t^\tau D_z^\zeta u \right)_{x_i x_i} + \mathbf{1}_{\{\zeta_i \geq 1\}} a_{ii}(z) \left( D_t^\tau D_z^{\zeta - e_i} u \right)_{x_i x_i} \\ &\quad + \mathbf{1}_{\{\zeta_i \geq 1\}} \sum_{\substack{\zeta' + \zeta'' = \zeta - e_i \\ |\zeta''| \leq |\zeta| - 2}} D_z^{\zeta'} a_{ii}(z) \left( D_t^\tau D_z^{\zeta''} u \right)_{x_i x_i} \\ &\quad + x_i \sum_{\substack{\zeta' + \zeta'' = \zeta \\ |\zeta'| \geq 1}} \left( D_z^{\zeta'} a_{ii} \right)(z) \left( D_t^\tau D_z^{\zeta''} u \right)_{x_i x_i}, \end{aligned} \quad (2.34)$$

where  $\zeta'$  and  $\zeta''$  are multi-indices in  $\mathbb{N}^{n+m}$ . We now estimate each term of the preceding inequality.

Applying [7, Inequality (5.62)], we have that

$$\begin{aligned} \|x_i a_{ii} D_t^\tau D_z^\zeta u_{x_i x_i}\|_{C_{WF}^\alpha([0, T] \times \bar{U})} &\leq \Lambda \|x_i D_t^\tau D_z^\zeta u_{x_i x_i}\|_{C_{WF}^\alpha([0, T] \times \bar{U})} \\ &\quad + K \|x_i D_t^\tau D_z^\zeta u_{x_i x_i}\|_{C([0, T] \times \bar{U})}, \end{aligned}$$

where we recall the definitions of the positive constants  $\Lambda$  and  $K$  in (2.32) and (3.2), respectively.

Because  $2\tau + |\zeta| \leq k$ , and we assume that  $u \in C_{WF}^{k, 2+\alpha}([0, T] \times \bar{U})$ , we have that

$$\|x_i D_t^\tau D_z^\zeta u_{x_i x_i}\|_{C_{WF}^\alpha([0, T] \times \bar{U})} \leq \|u\|_{C_{WF}^{k, 2+\alpha}([0, T] \times \bar{U})},$$

and applying the interpolation inequality (2.22), we also have that there are positive constants,  $C = C(\alpha, k, m, n, T)$  and  $m_k = m_k(\alpha, k, m, n)$ , such that

$$\|x_i D_t^\tau D_z^\zeta u_{x_i x_i}\|_{C([0, T] \times \bar{U})} \leq \varepsilon \|u\|_{C_{WF}^{k, 2+\alpha}([0, T] \times \bar{U})} + C\varepsilon^{-m_k} \|u\|_{C([0, T] \times \bar{U})}.$$

Combining the preceding three inequalities, we obtain that

$$\|x_i a_{ii} D_t^\tau D_z^\zeta u_{x_i x_i}\|_{C_{WF}^\alpha([0, T] \times \bar{U})} \leq (\Lambda + K\varepsilon) \|u\|_{C_{WF}^{k, 2+\alpha}([0, T] \times \bar{U})} + C\varepsilon^{-m_k} \|u\|_{C([0, T] \times \bar{U})}, \quad (2.35)$$

where now  $C = C(\alpha, k, K, m, n, T)$  is a positive constant.

Applying again [7, Inequality (5.62)], we have that

$$\begin{aligned} \|a_{ii} D_t^\tau D_z^{\zeta - e_i} u_{x_i x_i}\|_{C_{WF}^\alpha([0, T] \times \bar{U})} &\leq \Lambda \|D_t^\tau D_z^{\zeta - e_i} u_{x_i x_i}\|_{C_{WF}^\alpha([0, T] \times \bar{U})} \\ &\quad + K \|D_t^\tau D_z^{\zeta - e_i} u_{x_i x_i}\|_{C([0, T] \times \bar{U})}. \end{aligned}$$

Writing the function  $D_t^\tau D_z^{\zeta - e_i} u_{x_i x_i} = D_t^\tau D_z^\zeta u_{x_i x_i}$ , and using the fact that  $2\tau + |\zeta| \leq k$  and  $u \in C_{WF}^{k, 2+\alpha}([0, T] \times \bar{U})$ , the interpolation inequality (2.19) gives us

$$\|D_t^\tau D_z^\zeta u_{x_i x_i}\|_{C([0, T] \times \bar{U})} \leq \varepsilon \|u\|_{C_{WF}^{k, 2+\alpha}([0, T] \times \bar{U})} + C\varepsilon^{-m_k} \|u\|_{C([0, T] \times \bar{U})},$$

and so, we obtain

$$\|a_{ii} D_t^\tau D_z^{\zeta - e_i} u_{x_i x_i}\|_{C_{WF}^\alpha([0, T] \times \bar{U})} \leq (\Lambda + K\varepsilon) \|u\|_{C_{WF}^{k, 2+\alpha}([0, T] \times \bar{U})} + C\varepsilon^{-m_k} \|u\|_{C([0, T] \times \bar{U})}, \quad (2.36)$$

We now consider the case of multi-indices  $\zeta'$  and  $\zeta''$  in  $\mathbb{N}^{n+m}$ , such that  $\zeta' + \zeta'' = \zeta - e_i$  and  $|\zeta''| \leq |\zeta| - 2$ . Because  $2\tau + |\zeta| \leq k$ , then  $2\tau + |\zeta''| + 2 \leq k$ , and using the fact that  $u \in C_{WF}^{k, 2+\alpha}([0, T] \times \bar{U})$ , we may apply the interpolation inequality (2.18) to the function  $D_t^\tau D_z^{\zeta''} u_{x_i x_i}$ , together with [7, Inequality (5.62)], to obtain

$$\|D_z^{\zeta'} a_{ii} D_t^\tau D_z^{\zeta''} u_{x_i x_i}\|_{C_{WF}^\alpha([0, T] \times \bar{U})} \leq (\Lambda + K\varepsilon) \|u\|_{C_{WF}^{k, 2+\alpha}([0, T] \times \bar{U})} + C\varepsilon^{-m_k} \|u\|_{C([0, T] \times \bar{U})}, \quad (2.37)$$

It remains to consider the case of indices  $\zeta'$  and  $\zeta''$  in  $\mathbb{N}^{n+m}$ , such that  $\zeta' + \zeta'' = \zeta$  and  $|\zeta'| \geq 1$ . Then we have that  $2\tau + |\zeta''| + 2 \leq k + 1$ , and we may apply the interpolation inequalities (2.27) to the function  $D_t^\tau D_z^{\zeta''} u_{x_i x_i}$ , together with [7, Inequality (5.62)], to obtain

$$\begin{aligned} \|x_i D_z^{\zeta'} a_{ii} D_t^\tau D_z^{\zeta''} u_{x_i x_i}\|_{C_{WF}^\alpha([0,T] \times \bar{U})} &\leq (\Lambda + K\varepsilon) \|u\|_{C_{WF}^{k,2+\alpha}([0,T] \times \bar{U})} \\ &\quad + C\varepsilon^{-m_k} \|u\|_{C([0,T] \times \bar{U})}, \end{aligned} \quad (2.38)$$

Combining identity (2.34) with inequalities (2.35)-(2.38), we obtain that the  $C_{WF}^\alpha([0,T] \times \bar{U})$ -Hölder norm of the function  $D_t^\tau D_z^\zeta (x_i a_{ii} u_{x_i x_i})$  is controlled by the right-hand side of inequality (2.30). This completes the proof.  $\square$

We have another technical estimate used in the proof of Theorem 1.4.

**Lemma 2.7** (Estimate of  $\varphi Lu$ ). *Let  $\alpha \in (0,1)$ ,  $k \in \mathbb{N}$  and  $T > 0$ . Let  $I \subseteq \{1, \dots, n\}$  and let  $\varphi \in C_{WF}^{k,\alpha}(\bar{S}_{n,m})$  be a function with support in  $\bar{M}_I'$ . Suppose that the coefficients of the operator  $L$  satisfy property (3.2). Then there are positive constants,  $C = C(\alpha, \|\varphi\|_{C_{WF}^{k,\alpha}(\bar{S}_{n,m})}, k, K, m, n, T)$  and  $m_k = m_k(\alpha, k, m, n)$ , such that for all functions  $u \in C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})$  with support in  $[0,T] \times \bar{M}_I''$ , we have that*

$$\begin{aligned} \|\varphi Lu\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} &\leq \left( \Lambda \|\varphi\|_{C(\bar{S}_{n,m})} + C\varepsilon \right) \|u\|_{C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})} \\ &\quad + C\varepsilon^{-m_k} \|u\|_{C([0,T] \times \bar{S}_{n,m})}, \end{aligned} \quad (2.39)$$

where the positive constant  $\Lambda$  is given by (2.31) with the set  $\bar{U}$  replaced by  $\text{supp } \varphi$ .

*Proof.* Estimate (2.39) is a straightforward consequence of Lemma 2.5. We only need to change the coefficients of the operator  $L$  by multiplying them by the function  $\varphi$ . We omit the detailed proof.  $\square$

We now give the proof of the interpolation inequalities in the anisotropic Hölder spaces.

*Proof of Proposition 2.1.* We consider  $\eta \in (0,1)$  to be a suitably chosen constant during the proofs of each of inequalities (2.6)-(2.17). We divide the proof into several steps.

**Step 1** (Proof of inequality (2.6)). It is sufficient to show that inequality (2.6) holds for the seminorm  $[u]_{C_{WF}^\alpha([0,T] \times \bar{S}_{n,m})}$ , and for this purpose we only need to consider differences,  $u(P_1) - u(P_2)$ , where all except one of the coordinates of the points  $P_1, P_2 \in [0,T] \times \bar{S}_{n,m}$  are identical. We outline the proof when the  $x_i$ -coordinates of  $P_1$  and  $P_2$  differ, but the case of the  $t$ -coordinate and of the  $y_l$ -coordinates can be treated in a similar manner. Notice that from inequalities (2.3), we can find a positive constant,  $C$ , such that

$$\frac{|x_i^1 - x_i^2|}{\rho(P_1, P_2)} \leq C, \quad (2.40)$$

We consider two situations:  $|x_i^1 - x_i^2| \leq \eta$  and  $|x_i^1 - x_i^2| > \eta$ .

**Case 1** (Points with  $x_i$ -coordinates close together). Assuming that  $|x_i^1 - x_i^2| \leq \eta$ , we have

$$\begin{aligned} |u(P_1) - u(P_2)| &\leq |x_i^1 - x_i^2| \|u_{x_i}\|_{C([0,T] \times \bar{S}_{n,m})} \\ &\leq \eta \frac{|x_i^1 - x_i^2|}{\eta} \|u\|_{C_{WF}^{2+\alpha}([0,T] \times \bar{S}_{n,m})} \\ &\leq \eta \left( \frac{|x_i^1 - x_i^2|}{\eta} \right)^\alpha \|u\|_{C_{WF}^{2+\alpha}([0,T] \times \bar{S}_{n,m})} \\ &\leq \eta^{1-\alpha} \left( \frac{|x_i^1 - x_i^2|}{\rho(P_1, P_2)} \right)^\alpha \rho^\alpha(P_1, P_2) \|u\|_{C_{WF}^{2+\alpha}([0,T] \times \bar{S}_{n,m})}, \end{aligned}$$

and so, using inequality (2.40), there is a positive constant,  $C = C(\alpha)$ , such that

$$\frac{|u(P_1) - u(P_2)|}{\rho^\alpha(P_1, P_2)} \leq C \eta^{1-\alpha} \|u\|_{C_{WF}^{2+\alpha}([0,T] \times \bar{S}_{n,m})}, \quad (2.41)$$

which concludes this case.

**Case 2** (Points with  $x_i$ -coordinates farther apart). Assuming that  $|x_i^1 - x_i^2| > \eta$ , we have

$$1 < \left( \frac{|x_i^1 - x_i^2|}{\eta} \right)^\alpha = \eta^{-\alpha} \left( \frac{|x_i^1 - x_i^2|}{\rho(P_1, P_2)} \right)^\alpha \rho^\alpha(P_1, P_2),$$

from where it follows by inequality (2.40) that there is a positive constant,  $C = C(\alpha)$ , such that  $1 \leq C \eta^{-\alpha} \rho^\alpha(P_1, P_2)$ . We then obtain that

$$|u(P_1) - u(P_2)| \leq 2 \|u\|_{C([0,T] \times \bar{S}_{n,m})} \leq C \eta^{-\alpha} \rho^\alpha(P_1, P_2) \|u\|_{C([0,T] \times \bar{S}_{n,m})},$$

which is equivalent to

$$\frac{|u(P_1) - u(P_2)|}{\rho^\alpha(P_1, P_2)} \leq C \eta^{-\alpha} \|u\|_{C([0,T] \times \bar{S}_{n,m})}, \quad (2.42)$$

which concludes this case.

By combining inequalities (2.41) and (2.42), we obtain

$$[u]_{C_{WF}^\alpha([0,T] \times \bar{S}_{n,m})} \leq C \eta^{1-\alpha} \|u\|_{C_{WF}^{2+\alpha}([0,T] \times \bar{S}_{n,m})} + C \eta^{-\alpha} \|u\|_{C([0,T] \times \bar{S}_{n,m})}.$$

Since  $\varepsilon \in (0, 1)$ , we may choose  $\eta \in (0, 1)$  such that  $\varepsilon = C \eta^{1-\alpha}$ . The preceding inequality then gives (2.6).

Inequalities (2.7), (2.8) and (2.9) are proved using a similar method, and so, we only outline the proof of inequality (2.7).

**Step 2** (Proof of inequality (2.7)). Let  $P \in [0, T] \times \bar{S}_{n,m}$ . Then, for any  $\eta > 0$ , we have

$$\begin{aligned} |u_{x_i}(P)| &\leq |u_{x_i}(P) - \eta^{-1} (u(P + \eta e_i) - u(P))| + 2\eta^{-1} \|u\|_{C([0,T] \times \bar{S}_{n,m})} \\ &= |u_{x_i}(P) - u_{x_i}(P + \eta \theta e_i)| + 2\eta^{-1} \|u\|_{C([0,T] \times \bar{S}_{n,m})} \\ &= \frac{|u_{x_i}(P) - u_{x_i}(P + \eta \theta e_i)|}{\rho^\alpha(P, P + \eta \theta e_i)} \rho^\alpha(P, P + \eta \theta e_i) + 2\eta^{-1} \|u\|_{C([0,T] \times \bar{S}_{n,m})}, \end{aligned}$$

for some constant  $\theta \in [0, 1]$ . From inequality (2.3) and the fact that we choose  $\eta \in (0, 1)$ , we obtain that there is a positive constant,  $C$ , such that

$$\rho(P, P + \eta \theta e_i) \leq C \eta^{1/2}, \quad \forall P \in [0, T] \times \bar{S}_{n,m}, \quad (2.43)$$

which gives us

$$|u_{x_i}(P)| \leq \eta^{\alpha/2} [u_{x_i}]_{C_{WF}^\alpha([0,T] \times \bar{S}_{n,m})} + 2\eta^{-1} \|u\|_{C([0,T] \times \bar{S}_{n,m})}, \quad \forall P \in [0, T] \times \bar{S}_{n,m}.$$

Since  $\varepsilon \in (0, 1)$ , we may choose  $\eta \in (0, 1)$  such that  $\varepsilon = \eta^{\alpha/2}$ , and inequality (2.7) follows immediately from the preceding one.

For inequalities (2.10)-(2.17), we assume that the function  $u$  is supported in  $[0, T] \times \bar{M}_I''$ , for a set of indices  $I \subseteq \{1, \dots, n\}$ . From Remark 2.6, without loss of generality we may assume that  $I = \{1, 2, \dots, n\}$ , because otherwise the  $x_j$ -variables, for  $j \in I^c$ , can be treated as the  $y_l$ -variables, for all  $l = 1, \dots, m$ . That is, by relabeling the variables, we may replace the space  $S_{n,m}$  by  $S_{n',m'}$ , where  $n' = |I|$  and  $m' = m + n - |I|$ , so that the function  $u$  may be assumed to have support in  $[0, T] \times \bar{M}_{\{1, \dots, n'\}}''$ .

Note that it is sufficient to prove inequalities (2.10), (2.11) and (2.14), as inequalities (2.12) and (2.13) follow from (2.11) and (2.14), respectively. The proof of inequality (2.14) is very similar to the proofs of inequalities (2.10) and (2.11), but simpler, and so, we omit its detailed proof for brevity.

**Step 3** (Proof of inequalities (2.10) and (2.11)). For any point,  $P = (t, z) \in [0, T] \times \bar{S}_{n,m}$ , and  $\eta > 0$ , we have for all  $i \in I$  and  $j = 1, \dots, n$ ,

$$\begin{aligned} |u_{x_i x_j}(P)| &\leq |u_{x_i x_j}(P) - \eta^{-1} (u_{x_i}(P + \eta e_j) - u_{x_i}(P))| + \eta^{-1} (|u_{x_i}(P)| + |u_{x_i}(P + \eta e_j)|) \\ &\leq |u_{x_i x_j}(P) - u_{x_i x_j}(P + \theta \eta e_j)| + \eta^{-1} (|u_{x_i}(P)| + |u_{x_i}(P + \eta e_j)|), \end{aligned} \quad (2.44)$$

for some  $\theta \in [0, 1]$ . If  $j \in I^c$ , we have

$$\begin{aligned} |\sqrt{x_i} u_{x_i x_j}(P)| &\leq \frac{|\sqrt{x_i} u_{x_i x_j}(P) - \sqrt{x_i} u_{x_i x_j}(P + \theta \eta e_j)|}{\rho^\alpha(P, P + \theta \eta e_j)} \rho^\alpha(P, P + \theta \eta e_j) \\ &\quad + 2\eta^{-1} \|\sqrt{x_i} u_{x_i}\|_{C([0,T] \times \bar{S}_{n,m})} \\ &\leq C\eta^{\alpha/2} [\sqrt{x_i} u_{x_i x_j}]_{C_{WF}^\alpha([0,T] \times \bar{S}_{n,m})} + 2\eta^{-1} \|\sqrt{x_i} u_{x_i}\|_{C([0,T] \times \bar{S}_{n,m})} \quad (\text{by (2.43)}). \end{aligned}$$

For all  $\varepsilon \in (0, 1)$ , we may choose  $\eta \in (0, 1)$  such that  $\varepsilon = C\eta^{\alpha/2}$  in the preceding inequality. Combining the resulting inequality with (2.7), and using the fact that the function  $u$  has support in  $[0, T] \times \bar{M}_I''$ , and that the domain  $M_I''$  is bounded in the  $x_i$ -coordinate, for all  $i \in I$ , we see that estimate (2.11) holds for all  $j \in I^c$ .

Next, we consider the case when  $j \in I$ , that is, we want to prove inequality (2.10). For brevity, we denote  $P' = P + \theta \eta e_j$ . We consider two distinct cases depending on whether  $\eta < x'_j/2$  or  $\eta \geq x'_j/2$ .

**Case 1** (Points with  $x_j$ -coordinates small). Assuming that  $\eta < x'_j/2$ , we obtain by (2.44) that

$$\begin{aligned} |\sqrt{x_i x_j} u_{x_i x_j}(P)| &\leq \frac{|\sqrt{x_i x_j} u_{x_i x_j}(P) - \sqrt{x_i x'_j} u_{x_i x_j}(P')|}{\rho^\alpha(P, P')} \rho^\alpha(P, P') \\ &\quad + \left| \left( \sqrt{x_i x_j} - \sqrt{x_i x'_j} \right) u_{x_i x_j}(P') \right| + 2\eta^{-1} \|\sqrt{x_i x_j} u_{x_i}\|_{C([0,T] \times \bar{S}_{n,m})}. \end{aligned} \quad (2.45)$$



Using inequality (2.43) and the fact that  $|x'_j - x_j| \leq \eta$ , by definitions of the points  $P$  and  $P'$ , we have that

$$\begin{aligned} |\sqrt{x_i x_j} u_{x_i x_j}(P)| &\leq \eta^{\alpha/2} [\sqrt{x_i x_j} u_{x_i x_j}]_{C_{WF}^\alpha([0, T] \times \bar{S}_{n, m})} \\ &\quad + \sqrt{\frac{\eta}{x'_j}} \left| \sqrt{x_i x'_j} u_{x_i x_j}(P') \right| + 2\eta^{-1} \|\sqrt{x_i x_j} u_{x_i}\|_{C([0, T] \times \bar{S}_{n, m})}, \end{aligned}$$

which gives, by our assumption that  $\eta < x'_j/2$ ,

$$\begin{aligned} |\sqrt{x_i x_j} u_{x_i x_j}(P)| &\leq \eta^{\alpha/2} [\sqrt{x_i x_j} u_{x_i x_j}]_{C_{WF}^\alpha([0, T] \times \bar{S}_{n, m})} \\ &\quad + \frac{1}{\sqrt{2}} \|\sqrt{x_i x_j} u_{x_i x_j}\|_{C([0, T] \times \bar{S}_{n, m})} + 2\eta^{-1} \|\sqrt{x_i x_j} u_{x_i}\|_{C([0, T] \times \bar{S}_{n, m})}, \end{aligned} \quad (2.46)$$

which concludes this case.

**Case 2** (Points with  $x_j$ -coordinates large). We now assume that  $\eta \geq x'_j/2$ . Using the fact that  $x'_j = x_j + \theta\eta$ , for some  $\theta \in [0, 1]$ , we have that  $|x'_j - x_j| \leq x'_j$ . From [7, Lemma 5.2.5], it follows that  $\sqrt{x_i x_j} u_{x_i x_j}$  tends to 0, as  $x_j$  approaches 0, and so, we obtain

$$\begin{aligned} \left| \left( \sqrt{x_i x_j} - \sqrt{x_i x'_j} \right) u_{x_i x_j}(P') \right| &\leq \left| \sqrt{x_i x'_j} u_{x_i x_j}(P') \right| = \frac{\left| \sqrt{x_i x'_j} u_{x_i x_j}(P') - 0 \right|}{\rho^\alpha(P', P'')} \rho^\alpha(P', P'') \\ &\leq [\sqrt{x_i x_j} u_{x_i x_j}]_{C_{WF}^\alpha([0, T] \times \bar{S}_{n, m})} (2\eta)^{\alpha/2}, \end{aligned}$$

where  $P''$  be the projection of  $P'$  on the hyperplane  $\{x_j = 0\}$ , which gives us the inequality  $\rho(P', P'') \leq \sqrt{x'_j} \leq \sqrt{2\eta}$ , used in the second line of the preceding estimate. By inequality (2.45), we obtain that there is a positive constant,  $C = C(\alpha)$ , such that

$$|\sqrt{x_i x_j} u_{x_i x_j}(P)| \leq C\eta^{\alpha/2} [\sqrt{x_i x_j} u_{x_i x_j}]_{C_{WF}^\alpha([0, T] \times \bar{S}_{n, m})} + 2\eta^{-1} \|\sqrt{x_i x_j} u_{x_i}\|_{C([0, T] \times \bar{S}_{n, m})}, \quad (2.47)$$

which concludes this case.

Combining inequalities (2.46) and (2.47), we obtain, for all  $P \in [0, T] \times \bar{S}_{n, m}$ , that

$$\begin{aligned} |\sqrt{x_i x_j} u_{x_i x_j}(P)| &\leq \frac{1}{\sqrt{2}} \|\sqrt{x_i x_j} u_{x_i x_j}\|_{C([0, T] \times \bar{S}_{n, m})} \\ &\quad + C\eta^{\alpha/2} [\sqrt{x_i x_j} u_{x_i x_j}]_{C_{WF}^\alpha([0, T] \times \bar{S}_{n, m})} + 2\eta^{-1} \|\sqrt{x_i x_j} u_{x_i}\|_{C([0, T] \times \bar{S}_{n, m})}. \end{aligned}$$

Rearranging terms yields

$$\|\sqrt{x_i x_j} u_{x_i x_j}\|_{C([0, T] \times \bar{S}_{n, m})} \leq C\eta^{\alpha/2} [\sqrt{x_i x_j} u_{x_i x_j}]_{C_{WF}^\alpha([0, T] \times \bar{S}_{n, m})} + C\eta^{-1} \|\sqrt{x_i x_j} u_{x_i}\|_{C([0, T] \times \bar{S}_{n, m})}.$$

Since  $\varepsilon \in (0, 1)$ , we may choose  $\eta \in (0, 1)$  in the preceding inequality such that  $\varepsilon = C\eta^{\alpha/2}$ . To obtain inequality (2.10), we then use (2.7), together with the fact that the function  $u$  has support in  $[0, T] \times \bar{M}_I''$ , and so, the domain  $M_I''$  is bounded in the  $x_i$  and  $x_j$ -coordinates, for all  $i, j \in I$ . This concludes the proof of the interpolation inequality (2.10), for all  $j \in I$ .

It remains to give the proofs of the interpolation inequalities (2.15)-(2.17). The proofs of inequalities (2.16) and (2.17) are similar to that of inequality (2.15), but simpler, and so we only give the details of the proof of inequality (2.15).

**Step 4** (Proof of inequality (2.15)). From inequality (2.7), we see that it is sufficient to prove that estimate (2.15) holds for the Hölder seminorm  $[x_i u_{x_i}]_{C_{WF}^\alpha([0,T] \times \bar{S}_{n,m})}$ . As in the proof of inequality (2.6), it suffices to consider the differences  $x_i^1 u_{x_i}(P_1) - x_i^2 u_{x_i}(P_2)$ , where all except one of the coordinates of the points  $P_1, P_2 \in [0, T] \times \bar{S}_{n,m}$  are identical. First, we consider the case when only the  $x_i$ -coordinates of the points  $P_1$  and  $P_2$  differ.

**Case 1** (Points with  $x_i$ -coordinates close together). Assuming that  $|x_i^1 - x_i^2| \leq \eta$ , and using the mean value theorem, there is a point  $P^*$  on the line segment connecting  $P_1$  and  $P_2$  such that

$$x_i^1 u_{x_i}(P_1) - x_i^2 u_{x_i}(P_2) = (x_i^* u_{x_i x_i}(P^*) + u_{x_i}(P^*)) (x_i^1 - x_i^2).$$

The argument used to prove Case 1 of Step 1 applies immediately to this setting, with the role of function  $u_{x_i}$  replaced by  $x_i u_{x_i x_i} + u_{x_i}$ , and we obtain that there is a positive constant,  $C = C(\alpha)$ , such that

$$\frac{|x_i^1 u_{x_i}(P_1) - x_i^2 u_{x_i}(P_2)|}{\rho^\alpha(P_1, P_2)} \leq C \eta^{1-\alpha} \|u\|_{C_{WF}^{2+\alpha}([0,T] \times \bar{S}_{n,m})}, \quad (2.48)$$

which concludes this case.

**Case 2** (Points with  $x_i$ -coordinates farther apart). Assuming that  $|x_i^1 - x_i^2| > \eta$ , the argument of Case 2 in Step 1 applies with  $u$  replaced by  $x_i u_{x_i}$ , and we obtain

$$\frac{|x_i^1 u_{x_i}(P_1) - x_i^2 u_{x_i}(P_2)|}{\rho^\alpha(P_1, P_2)} \leq C \eta^{-\alpha} \|x_i u_{x_i}\|_{C([0,T] \times \bar{S}_{n,m})}.$$

Since  $\varepsilon \in (0, 1)$ , we may choose  $\eta$  such that  $\varepsilon = \eta^{\alpha+1}$  in inequality (2.7), and we obtain

$$\frac{|x_i^1 u_{x_i}(P_1) - x_i^2 u_{x_i}(P_2)|}{\rho^\alpha(P_1, P_2)} \leq C \eta \|u\|_{C_{WF}^{2+\alpha}([0,T] \times \bar{S}_{n,m})} + C \eta^{-m_0(1+\alpha)-\alpha} \|u\|_{C([0,T] \times \bar{S}_{n,m})}, \quad (2.49)$$

which concludes this case.

Combining inequalities (2.48) and (2.49) gives us that

$$\frac{|x_i^1 u_{x_i}(P_1) - x_i^2 u_{x_i}(P_2)|}{\rho^\alpha(P_1, P_2)} \leq C \eta^{1-\alpha} \|u\|_{C_{WF}^{2+\alpha}([0,T] \times \bar{S}_{n,m})} + C \eta^{-m_0(1+\alpha)-\alpha} \|u\|_{C([0,T] \times \bar{S}_{n,m})}. \quad (2.50)$$

We now consider the case when only the  $x_j$ -coordinates, with  $j \neq i$  and  $j \in I$ , differ. Let  $x_j^k$  be the  $x_j$ -coordinates of the points  $P_k$ , for  $k = 1, 2$ , and assume that  $x_j^1 < x_j^2$ . We have that

$$\begin{aligned} \sqrt{x_i} (u_{x_i}(P_2) - u_{x_i}(P_1)) &= \sqrt{x_i} \int_0^{x_j^2 - x_j^1} u_{x_i x_j}(P_1 + t e_j) dt \\ &= \sqrt{x_i} \int_0^{x_j^2 - x_j^1} \sqrt{x_j^1 + t} u_{x_i x_j}(P_1 + t e_j) \frac{1}{\sqrt{x_j^1 + t}} dt, \end{aligned}$$

which gives us that

$$\sqrt{x_i} |u_{x_j}(P_2) - u_{x_j}(P_1)| \leq 2 \|\sqrt{x_i x_j} u_{x_i x_j}\|_{C([0,T] \times \bar{S}_{n,m})} \left( \sqrt{x_j^2} - \sqrt{x_j^1} \right).$$

Because the function  $u$  has support in  $[0, T] \times \bar{M}_I''$  and  $j \in I$ , we obtain from property (2.3) of the distance function  $\rho$  that there is a positive constant,  $C$ , such that

$$\frac{\sqrt{x_i} |u_{x_j}(P_2) - u_{x_j}(P_1)|}{\rho^\alpha(P_1, P_2)} \leq C \|\sqrt{x_i x_j} u_{x_i x_j}\|_{C([0,T] \times \bar{S}_{n,m})} \left( \sqrt{x_j^2} - \sqrt{x_j^1} \right)^{1-\alpha}.$$

Using the fact that  $u$  has support in  $[0, T] \times \bar{M}_I''$ , and that the set  $M_I''$  is bounded in the  $x_i$ - and  $x_j$ -directions, from identity (2.5), it follows that there is a positive constant,  $C = C(\alpha)$ , such that

$$\frac{|x_i(u_{x_j}(P_2) - u_{x_j}(P_1))|}{\rho^\alpha(P_1, P_2)} \leq C \|\sqrt{x_i x_j} u_{x_i x_j}\|_{C([0, T] \times \bar{S}_{n, m})}.$$

From the interpolation inequality (2.10), it follows that there are positive constants,  $C = C(\alpha, m, n, T)$  and  $m_0 = m_0(\alpha, m, n)$ , such that for all  $\varepsilon \in (0, 1)$  we have that

$$\frac{|x_i(u_{x_j}(P_2) - u_{x_j}(P_1))|}{\rho^\alpha(P_1, P_2)} \leq \varepsilon \|u\|_{C_{WF}^{2+\alpha}([0, T] \times \bar{S}_{n, m})} + C\varepsilon^{-m_0} \|u\|_{C([0, T] \times \bar{S}_{n, m})}. \quad (2.51)$$

A similar argument applied when only the  $x_j$ -coordinates, with  $j \neq i$  and  $j \in I^c$ , or only the  $y_l$ -coordinates of the points  $P_1$  and  $P_2$ , with  $l = 1, \dots, m$ , differ also yields the analogous inequality of (2.51).

It remains to consider the case when only the  $t$ -coordinates of the points  $P_1$  and  $P_2$  differ. We denote  $P_k = (t_k, z)$ , for  $k = 1, 2$ , and we let  $\gamma := \sqrt{|t_1 - t_2|}$ .

**Case 3** (Points with  $t$ -coordinates close together). Assuming that  $|t_1 - t_2| < \eta$ , we have

$$\begin{aligned} |u_{x_i}(P_1) - u_{x_i}(P_2)| &\leq \left| u_{x_i}(t_1, z) - \frac{1}{\gamma} (u(t_1, z + \gamma e_i) - u(t_1, z)) \right| \\ &\quad + \left| u_{x_i}(t_2, z) - \frac{1}{\gamma} (u(t_2, z + \gamma e_i) - u(t_2, z)) \right| \\ &\quad + \frac{1}{\gamma} |u(t_1, z + \gamma e_i) - u(t_2, z + \gamma e_i)| + \frac{1}{\gamma} |u(t_1, z) - u(t_2, z)|. \end{aligned}$$

and using the mean value theorem, there are points  $t_k^* \in [0, T]$  and  $P_k^* \in \bar{S}_{n, m}$ , for  $k = 1, 2$ , such that

$$\begin{aligned} |u_{x_i}(P_1) - u_{x_i}(P_2)| &= |u_{x_i}(t_1, z) - u_{x_i}(t_1, z + \theta_1 \gamma e_i)| + |u_{x_i}(t_2, z) - u_{x_i}(t_2, z + \theta_2 \gamma e_i)| \\ &\quad + \frac{|t_1 - t_2|}{\gamma} |u_t(t_1^*, z + \gamma e_i)| + \frac{|t_1 - t_2|}{\gamma} |u_t(t_2^*, z)| \\ &\leq |u_{x_i x_i}(t_1, P_1^*)| \gamma + |u_{x_i x_i}(t_2, P_2^*)| \gamma \\ &\quad + \frac{|t_1 - t_2|}{\gamma} |u_t(t_1^*, z + \gamma e_i)| + \frac{|t_1 - t_2|}{\gamma} |u_t(t_2^*, z)|. \end{aligned}$$

Notice that  $\rho(P_1, P_2) = \sqrt{|t_1 - t_2|} = \gamma$  and so, by multiplying the preceding inequality by  $x_i$ , and using the fact that  $u$  has support in  $[0, T] \times \bar{M}_I''$ , and that  $M_I''$  is bounded in the  $x_i$ -coordinate, for all  $i \in I$ , we obtain

$$\begin{aligned} \frac{|x_i u_{x_i}(P_1) - x_i u_{x_i}(P_2)|}{\rho^\alpha(P_1, P_2)} &\leq 2 \|x_i u_{x_i x_i}\|_{C([0, T] \times \bar{S}_{n, m})} |t_1 - t_2|^{\frac{1-\alpha}{2}} \\ &\quad + 2 |t_1 - t_2|^{1-\frac{1+\alpha}{2}} \|x_i u_t\|_{C([0, T] \times \bar{S}_{n, m})}, \end{aligned}$$

and thus, there is a positive constant,  $C$ , such that

$$\frac{|x_i u_{x_i}(P_1) - x_i u_{x_i}(P_2)|}{\rho^\alpha(P_1, P_2)} \leq C \eta^{\frac{1-\alpha}{2}} \|u\|_{C_{WF}^{2+\alpha}([0, T] \times \bar{S}_{n, m})}. \quad (2.52)$$

**Case 4** (Points with  $t$ -coordinates farther apart). Assuming that  $|t_1 - t_2| \geq \eta$ , it immediately follows that

$$\frac{|x_i u_{x_i}(P_1) - x_i u_{x_i}(P_2)|}{\rho^\alpha(P_1, P_2)} \leq 2 \eta^{-\frac{\alpha}{2}} \|x_i u_{x_i}\|_{C([0, T] \times \bar{S}_{n, m})}, \quad (2.53)$$

which concludes this case.

By combining inequalities (2.52) and (2.53), we obtain

$$\frac{|x_i^1 u_{x_i}(P_1) - x_i^2 u_{x_i}(P_2)|}{\rho^\alpha(P_1, P_2)} \leq C \eta^{\frac{1-\alpha}{2}} \|u\|_{C_{WF}^{2+\alpha}([0,T] \times \bar{S}_{n,m})} + 2\eta^{-\frac{\alpha}{2}} \|u_{x_i}\|_{C([0,T] \times \bar{S}_{n,m})}. \quad (2.54)$$

Combining inequalities (2.50), (2.51) and (2.54) it follows that

$$\begin{aligned} \frac{|x_i^1 u_{x_i}(P_1) - x_i^2 u_{x_i}(P_2)|}{\rho^\alpha(P_1, P_2)} &\leq C (\eta^{1-\alpha} + \varepsilon) \|u\|_{C_{WF}^{2+\alpha}([0,T] \times \bar{S}_{n,m})} \\ &\quad + 2\eta^{-\frac{\alpha}{2}} \|u_{x_i}\|_{C([0,T] \times \bar{S}_{n,m})} + C \left( \eta^{-m_0(1+\alpha)-\alpha} + \varepsilon^{-m_0} \right) \|u\|_{C([0,T] \times \bar{S}_{n,m})}. \end{aligned}$$

Choosing  $\eta = \eta(\varepsilon) \in (0, 1)$  small enough and using inequality (2.7), we immediately obtain the interpolation inequality (2.15). This concludes the proof of Step 4.

This completes the proof of Proposition 2.1.  $\square$

### 3. LOCAL A PRIORI SCHAUDER ESTIMATES

In this section we give the proofs of Theorems 1.1 and 1.2. Our proof is based on a localization procedure of N.V. Krylov used in the proof of [17, Theorem 8.11.1], the interpolation inequalities for anisotropic Hölder spaces in Corollary 2.4, and the global a priori Schauder estimates obtained in [7, Theorem 10.0.2]. We begin with stating the properties of the coefficients of the differential operator  $L$ .

**Assumption 3.1** (Coefficients). There is a nonnegative integer,  $k$ , and positive constants,  $\delta$  and  $K$ , such that

1. The second-order coefficient functions satisfy the *strict ellipticity condition*: for all sets of indices,  $I \subseteq \{1, \dots, n\}$ , for all  $z \in \bar{M}_I$ ,  $\xi \in \mathbb{R}^n$  and  $\eta \in \mathbb{R}^m$ , we have

$$\begin{aligned} &\sum_{i \in I} a_{ii}(z) \xi_i^2 + \sum_{i \in I^c} x_i a_{ii}(z) \xi_i^2 + \sum_{i,j \in I} \tilde{a}_{ij}(z) \xi_i \xi_j + \sum_{i \in I} \sum_{j \in I^c} x_j (\tilde{a}_{ij}(z) + \tilde{a}_{ji}(z)) \xi_i \xi_j \\ &+ \sum_{i,j \in I^c} x_i x_j \tilde{a}_{ij}(z) \xi_i \xi_j + \sum_{i \in I} \sum_{l=1}^m c_{il}(z) \xi_i \eta_l + \sum_{i \in I^c} \sum_{l=1}^m x_i c_{il}(z) \xi_i \eta_l + \sum_{k,l=1}^m d_{kl}(z) \eta_k \eta_l \\ &\geq \delta (|\xi|^2 + |\eta|^2). \end{aligned} \quad (3.1)$$

2. The coefficient functions are *Hölder continuous*: for all sets of indices,  $I \subseteq \{1, \dots, n\}$ , we have

$$\begin{aligned} &\sum_{i \in I} \|a_{ii}\|_{C_{WF}^{k,\alpha}(\bar{M}_I)} + \sum_{i \in I^c} \|x_i a_{ii}\|_{C_{WF}^{k,\alpha}(\bar{M}_I)} + \sum_{i,j \in I} \|\tilde{a}_{ij}\|_{C_{WF}^{k,\alpha}(\bar{M}_I)} \\ &+ \sum_{i \in I} \sum_{j \in I^c} \left( \|x_j \tilde{a}_{ij}\|_{C_{WF}^{k,\alpha}(\bar{M}_I)} + \|x_j \tilde{a}_{ji}\|_{C_{WF}^{k,\alpha}(\bar{M}_I)} \right) + \sum_{i,j \in I^c} \|x_i x_j \tilde{a}_{ij}(z)\|_{C_{WF}^{k,\alpha}(\bar{M}_I)} \\ &+ \sum_{i \in I} \sum_{l=1}^m \|c_{il}\|_{C_{WF}^{k,\alpha}(\bar{M}_I)} + \sum_{i \in I^c} \sum_{l=1}^m \|x_i c_{il}\|_{C_{WF}^{k,\alpha}(\bar{M}_I)} + \sum_{k,l=1}^m \|d_{kl}\|_{C_{WF}^{k,\alpha}(\bar{M}_I)} \\ &+ \sum_{i=1}^n \|b_i\|_{C_{WF}^{k,\alpha}(\bar{M}_I)} + \sum_{k=1}^m \|e_k\|_{C_{WF}^{k,\alpha}(\bar{M}_I)} \\ &\leq K. \end{aligned} \quad (3.2)$$

3. The drift coefficient functions satisfy the *nonnegativity condition*:

$$b_i(z) \geq 0 \quad \text{on } \{z = (x, y) \in \partial S_{n,m} : x_i = 0\}, \quad \forall i = 1, \dots, n. \quad (3.3)$$

We can now give the

*Proof of Theorem 1.1.* Remark 2.6 applies to the set  $M'_I$  in place of  $M''_I$ , defined in (2.4) and (2.5), respectively. Thus, without loss of generality we may assume that  $B_{2r}(z^0) \subset M'_{\{1, \dots, n\}}$ , when  $r > 0$  is chosen small enough. Let  $L_0$  be defined similarly to the differential operator  $L$ , but with coefficients replaced by their values at  $z^0$ , that is,

$$\begin{aligned} L_0 u = & \sum_{i=1}^n (x_i a_{ii}(z^0) u_{x_i x_i} + b_i(z^0) u_{x_i}) + \sum_{i,j=1}^n x_i x_j \tilde{a}_{ij}(z^0) u_{x_i x_j} \\ & + \sum_{i=1}^n \sum_{l=1}^m x_i c_{il}(z^0) u_{x_i y_l} + \sum_{k,l=1}^m d_{kl}(z^0) u_{y_k y_l} + \sum_{l=1}^m e_l(z^0) u_{y_l}. \end{aligned}$$

Let  $\varphi : \mathbb{R} \rightarrow [0, 1]$  be a smooth function such that  $\varphi(t) = 0$  for  $t < 0$ , and  $\varphi(t) = 1$  for  $t > 1$ . Let

$$r_N = r \sum_{i=0}^N \frac{1}{2^i}, \quad \forall N \in \mathbb{N},$$

and consider the sequence of smooth cut-off functions,  $\{\eta_N\}_{N \geq 1} \subset C_c^\infty(\bar{S}_{n,m})$ , defined by

$$\eta_N(z) := \varphi\left(\frac{r_{N+1} - |z - z^0|}{r_{N+1} - r_N}\right), \quad \forall z \in \bar{S}_{n,m}, \quad \forall N \in \mathbb{N}.$$

We also let,

$$T_N = \frac{T_0}{2} + \frac{T_0}{2} \left(2 - \sum_{i=0}^N \frac{1}{2^i}\right), \quad \forall N \in \mathbb{N},$$

and consider the sequence of smooth cut-off functions,  $\{\psi_N\}_{N \geq 1} \subset C_c^\infty([0, \infty))$ , defined by

$$\psi_N(t) := \varphi\left(\frac{T_{N+1} - t}{T_{N+1} - T_N}\right), \quad \forall t \in [0, \infty), \quad \forall N \in \mathbb{N}.$$

We set  $\varphi_N(t, z) := \psi_N(t) \eta_N(z)$ , for all  $(t, z) \in [0, T] \times \bar{S}_{n,m}$ , and we let  $Q_N := [T_N, T] \times B_{r_N}(z^0)$ . Then, we see that  $0 \leq \varphi_N \leq 1$ , and  $\varphi_N \equiv 1$  on  $Q_N$ , and  $\varphi_N \equiv 0$  on  $Q_{N+1}^c$ , where  $Q_{N+1}^c$  denotes the complement of  $Q_{N+1}$  in  $[0, T] \times \bar{S}_{n,m}$ . By [7, Lemma 10.1.3], we can find a positive constant,  $c = c(k, m, n, \varphi)$ , such that for all  $N \in \mathbb{N}$ , we have that

$$\|D_t^\tau D_z^\zeta \varphi_N\|_{C_{WF}^\alpha([0, T] \times \bar{S}_{n,m})} \leq c \rho^N \left(r^{-(k+3)} + T_0^{-(k+3)}\right), \quad (3.4)$$

for all  $\tau \in \mathbb{N}$ ,  $\zeta \in \mathbb{N}^{n+m}$ , such that  $2\tau + |\zeta| = k + 2$ , where  $\rho := 2^{k+3} > 1$ . We let

$$\alpha_N := \|u \varphi_N\|_{C_{WF}^{k, 2+\alpha}([0, T] \times \bar{S}_{n,m})}, \quad \forall N \in \mathbb{N}. \quad (3.5)$$

We may assume that  $B_{2r}(z^0)$  is included in a compact manifold with corners,  $P$  (see [7, §2] for the definition of compact manifolds with corners). We extend the operator  $L_0$  from  $\bar{B}_{2r}(z^0)$  to  $P$  such that it satisfies the hypotheses of [7, Theorem 10.0.2], and we obtain that there is a positive constant,  $C = C(\alpha, \delta, k, K, m, n, T)$ , such that

$$\begin{aligned} \alpha_N & \leq C \|(u \varphi_N)_t - L_0(u \varphi_N)\|_{C_{WF}^{k, \alpha}([0, T] \times \bar{S}_{n,m})} \\ & \leq C \|(u \varphi_N)_t - L(u \varphi_N)\|_{C_{WF}^{k, \alpha}([0, T] \times \bar{S}_{n,m})} + C \|(L_0 - L)(u \varphi_N)\|_{C_{WF}^{k, \alpha}([0, T] \times \bar{S}_{n,m})}. \end{aligned} \quad (3.6)$$

We apply Lemma 2.5 to the function  $u\varphi_N$ , the set  $U = B_{r_N}(z_0)$ , and with  $L$  replaced by the operator  $L_0 - L$ , to estimate the last term in the preceding inequality. Notice that because the coefficients of the operator  $L$  are assumed to belong to  $C_{WF}^\alpha([0, T] \times \bar{S}_{n,m})$ , the constant  $\Lambda$  in (2.31) satisfies the bound  $\Lambda \leq Cr^{\alpha/2}$ , where  $C = C(K, m, n)$ . It then follows that there are positive constants,  $C = C(\alpha, k, K, m, n)$  and  $m_k = m_k(\alpha, k, m, n)$ , such that for all  $\varepsilon \in (0, 1)$ , we have

$$\begin{aligned} \|(L_0 - L)(u\varphi_N)\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} &\leq C(r^{\alpha/2} + \varepsilon)\|u\varphi_N\|_{C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})} \\ &\quad + C\varepsilon^{-m_k}\|u\varphi_N\|_{C([0,T] \times \bar{S}_{n,m})}. \end{aligned} \quad (3.7)$$

We now estimate the first term on the right-hand side of inequality (3.6). We have that

$$(u\varphi_N)_t - L(u\varphi_N) = \varphi_N(u_t - Lu) + u(\varphi_N)_t - [L, \varphi_N]u, \quad (3.8)$$

where the last term is given by

$$\begin{aligned} [L, \varphi_N]u &= uL\varphi_N + 2 \sum_{i=1}^n x_i a_{ii} u_{x_i}(\varphi_N)_{x_i} + \sum_{i,j=1}^n x_i x_j \tilde{a}_{ij} (u_{x_i}(\varphi_N)_{x_j} + u_{x_j}(\varphi_N)_{x_i}) \\ &\quad + \sum_{i=1}^n \sum_{l=1}^m x_i c_{il} (u_{x_i}(\varphi_N)_{y_l} + u_{y_l}(\varphi_N)_{x_i}) + \sum_{l,k=1}^m d_{lk} (u_{y_l}(\varphi_N)_{y_k} + u_{y_k}(\varphi_N)_{y_l}). \end{aligned} \quad (3.9)$$

Using [7, Inequality (5.62)], together with the fact that the support of the function  $\varphi_N$  is included in the set  $Q_{N+1}$ , we have that

$$\|\varphi_N(u_t - Lu)\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} \leq 2\|\varphi_N\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} \|u_t - Lu\|_{C_{WF}^{k,\alpha}(\bar{Q}_{N+1})},$$

and estimate (3.4) gives that there is a positive constant,  $c = c(k, m, n, \varphi)$ , such that

$$\|\varphi_N(u_t - Lu)\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} \leq c\rho^N \left( r^{-(k+3)} + T_0^{-(k+3)} \right) \|u_t - Lu\|_{C_{WF}^{k,\alpha}([T_0/2, T] \times \bar{B}_{2r}(z_0))}. \quad (3.10)$$

Because the coefficients of the operator  $L$  satisfy inequality (3.2), using estimate (3.4), we can find a positive constant,  $C = C(\alpha, k, K, m, n)$ , such that

$$\begin{aligned} &\|u(\varphi_N)_t - [L, \varphi_N]u\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} \\ &\leq C\rho^N \left( r^{-(k+3)} + T_0^{-(k+3)} \right) \left( \|u\varphi_{N+1}\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} \right. \\ &\quad \left. + \sum_{i=1}^n \|x_i(u\varphi_{N+1})_{x_i}\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} + \sum_{l=1}^m \|(u\varphi_{N+1})_{y_l}\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} \right). \end{aligned}$$

The interpolation inequalities (2.18), (2.27), and (2.29), together with the preceding inequality, equality (3.8) and estimate (3.10) give us, for all  $\varepsilon \in (0, 1)$ ,

$$\begin{aligned} &\|(u\varphi_N)_t - L(u\varphi_N)\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} \\ &\leq C\rho^N \left( r^{-(k+3)} + T_0^{-(k+3)} \right) \left( \|u_t - Lu\|_{C_{WF}^{k,\alpha}([T_0/2, T] \times \bar{B}_{2r}(z_0))} \right. \\ &\quad \left. + \varepsilon \|u\varphi_{N+1}\|_{C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})} + \varepsilon^{-m_k} \|u\varphi_{N+1}\|_{C([0,T] \times \bar{S}_{n,m})} \right), \end{aligned} \quad (3.11)$$



where  $C = C(\alpha, k, K, m, n, T)$  is a positive constant. Combining inequalities (3.6), (3.7) and (3.11), it follows that

$$\begin{aligned} \alpha_N &\leq C \left( \varepsilon \rho^N \left( r^{-(k+3)} + T_0^{-(k+3)} \right) + r^{\alpha/2} + \varepsilon \right) \alpha_{N+1} \\ &\quad + C \rho^N \left( r^{-(k+3)} + T_0^{-(k+3)} \right) \|u_t - Lu\|_{C_{WF}^{k,\alpha}([T_0/2, T] \times \bar{B}_{2r}(z^0))} \\ &\quad + C \rho^N \left( r^{-(k+3)} + T_0^{-(k+3)} \right) \varepsilon^{-m_k} \|u\varphi_{N+1}\|_{C([0, T] \times \bar{S}_{n,m})}, \end{aligned} \quad (3.12)$$

where  $C = C(\alpha, \delta, k, K, m, n, T)$  is a positive constant. Let  $\gamma \in (0, 1)$  be chosen such that

$$\rho^{1+m_k} \gamma \leq \frac{1}{2}, \quad (3.13)$$

Let  $r_0 = r_0(\alpha, k, m, n)$  be a positive constant such that

$$C r_0^{\alpha/2} \leq \frac{\gamma}{3},$$

and given any  $r \in (0, r_0)$  and  $T_0 \in (0, T)$ , choose  $\varepsilon = \varepsilon(r, T_0) \in (0, 1)$  such that

$$C \varepsilon \rho^N \left( r^{-(k+3)} + T_0^{-(k+3)} \right) = \frac{\gamma}{3}, \quad \text{and} \quad C \varepsilon \leq \frac{\gamma}{3}.$$

Then we can rewrite inequality (3.12) in the form

$$\begin{aligned} \alpha_N &\leq \gamma \alpha_{N+1} + C \rho^N \left( r^{-(k+3)} + T_0^{-(k+3)} \right) \|u_t - Lu\|_{C_{WF}^{k,\alpha}([T_0/2, T] \times \bar{B}_{2r}(z^0))} \\ &\quad + (3C)^{1+m_k} \left( r^{-(k+3)} + T_0^{-(k+3)} \right)^{1+m_k} \gamma^{-m_k} \rho^{(1+m_k)N} \|u\varphi_{N+1}\|_{C([0, T] \times \bar{S}_{n,m})}. \end{aligned}$$

We multiply the preceding inequality by  $\gamma^N$ , and we let

$$C_1 := \max \left\{ C \left( r^{-(k+3)} + T_0^{-(k+3)} \right), (3C)^{1+m_k} \left( r^{-(k+3)} + T_0^{-(k+3)} \right)^{1+m_k} \gamma^{-m_k} \right\}.$$

Then using inequality (3.13), we obtain for all  $r \in (0, r_0)$ ,

$$\gamma^N \alpha_N \leq \gamma^{N+1} \alpha_{N+1} + \frac{C_1}{2^N} \left( \|u_t - Lu\|_{C_{WF}^{k,\alpha}([T_0/2, T] \times \bar{B}_{2r}(z^0))} + \|u\|_{C([T_0/2, T] \times \bar{B}_{2R}(z^0))} \right).$$

Summing the terms of the preceding inequality yields

$$\sum_{N=0}^{\infty} \gamma^N \alpha_N \leq \sum_{N=0}^{\infty} \gamma^{N+1} \alpha_{N+1} + 2C_1 \left( \|u_t - Lu\|_{C_{WF}^{k,\alpha}([T_0/2, T] \times \bar{B}_{2r}(z^0))} + \|u\|_{C([T_0/2, T] \times \bar{B}_{2R}(z^0))} \right).$$

The sum  $\sum_{N=0}^{\infty} \gamma^N \alpha_N$  is well-defined because we assumed that the function  $u$  belongs to the space of functions  $C_{WF}^{k, 2+\alpha}([T_0/2, T] \times \bar{B}_{2r}(z^0))$ , while  $\gamma \in (0, 1)$ . By subtracting the term  $\sum_{N=1}^{\infty} \gamma^N \alpha_N$  from both sides of the preceding inequality, we obtain the desired inequality (1.3).  $\square$

*Proof of Theorem 1.2.* We can use the same argument to prove Theorem 1.2 that we used in the proof of Theorem 1.1, with the only modification that the function  $\psi_N$  is no longer needed, and  $T_N$  is chosen to be 0, for all  $N \in \mathbb{N}$ .  $\square$

## 4. EXISTENCE AND UNIQUENESS OF SOLUTIONS

In this section, we give the proofs of Theorems 1.4 and 1.5. The proof of Theorem 1.4 relies on the a priori Schauder estimates established in [7, Theorem 10.0.2], and the supremum estimates derived from [7, Proposition 3.3.1]. Finally, the existence result described in Theorem 1.5, for continuous initial data, is based on Theorem 1.4 and a compactness argument which uses the a priori Schauder estimates in Theorem 1.1.

We begin with

**Proposition 4.1** (Comparison principle). *Let  $T > 0$ , and assume that the coefficients of the differential operator  $L$  have the property that*

$$a_{ii}, \tilde{a}_{ij}, b_i, c_{il}, d_{lk}, e_l \in C([0, T] \times \bar{S}_{n,m}),$$

*for all  $i, j = 1, \dots, n$  and all  $l, k = 1, \dots, m$ . Let  $u$  be a function such that*

$$u \in C([0, T] \times \bar{S}_{n,m}) \cap C^1((0, T] \times \bar{S}_{n,m}) \cap C^2((0, T] \times S_{n,m}),$$

*and such that*

$$\lim_{x_i \downarrow 0} x_i u_{x_i x_i}(t, z) = 0, \quad \forall t \in (0, T], \quad \forall i = 1, \dots, n.$$

*Assume that  $u$  satisfies*

$$\begin{aligned} u_t - Lu &\leq 0 \quad \text{on } (0, T] \times \bar{S}_{n,m}, \\ u(0, \cdot) &\leq 0 \quad \text{on } S_{n,m}. \end{aligned}$$

*Then, we have that  $u \leq 0$  on  $[0, T] \times \bar{S}_{n,m}$ .*

*Proof.* The proof is very similar to that of [7, Proposition 3.1.1], and so, we omit its detailed proof for brevity.  $\square$

We have the following consequence of Proposition 4.1.

**Corollary 4.2** (Maximum principle). *Assume that the hypotheses of Proposition 4.1 hold, and that  $u$  is a solution to the inhomogeneous initial-value problem (1.2), where  $f \in C(\bar{S}_{n,m})$  and  $g \in C([0, T] \times \bar{S}_{n,m})$ . Then*

$$\|u\|_{C([0, T] \times \bar{S}_{n,m})} \leq \|f\|_{C(\bar{S}_{n,m})} + T\|g\|_{C([0, T] \times \bar{S}_{n,m})}. \quad (4.1)$$

*Proof.* We consider the auxiliary function,

$$v(t, z) := \|f\|_{C([0, T] \times \bar{S}_{n,m})} + t\|g\|_{C([0, T] \times \bar{S}_{n,m})}, \quad \forall (t, z) \in [0, T] \times \bar{S}_{n,m},$$

and we apply Proposition 4.1 to  $\pm u - v$ . The supremum estimate (4.1) follows immediately.  $\square$

We can now give the

*Proof of Theorem 1.4.* Uniqueness of solutions is a straightforward consequence of Proposition 4.1, and so, we only consider the question of existence of solutions. The proof of existence employs the method used in proving existence of solutions to parabolic partial differential equations outlined in [3, Theorem II.1.1]. We let  $\widehat{C}_{WF}^{k, 2+\alpha}([0, T] \times \bar{S}_{n,m})$  denote the Banach space of functions  $u \in C_{WF}^{k, 2+\alpha}([0, T] \times \bar{S}_{n,m})$  such that  $u(0, \cdot) \equiv 0$  on  $\bar{S}_{n,m}$ . Without loss of generality we may assume  $f \equiv 0$  in the initial-value problem (1.2), because the function  $Lf \in C_{WF}^{k, \alpha}(\bar{S}_{n,m})$ , when Assumption 3.1 holds and  $f \in C_{WF}^{k, 2+\alpha}([0, T] \times \bar{S}_{n,m})$ . We also have that

$$\partial_t - L : \widehat{C}_{WF}^{k, 2+\alpha}([0, T] \times \bar{S}_{n,m}) \rightarrow C_{WF}^{k, \alpha}([0, T] \times \bar{S}_{n,m})$$

is a well-defined operator. Our goal is to show that  $\partial_t - L$  is invertible and we accomplish this by constructing a bounded linear operator,  $V : C_{WF}^{k,\alpha}([0, T] \times \bar{S}_{n,m}) \rightarrow \widehat{C}_{WF}^{k,2+\alpha}([0, T] \times \bar{S}_{n,m})$ , such that

$$\left\| (\partial_t - L)V - I_{C_{WF}^{k,\alpha}([0, T] \times \bar{S}_{n,m})} \right\| < 1. \quad (4.2)$$

For this purpose, we fix  $r > 0$  and we choose a sequence of points,  $\{z^N\}_{N \geq 1}$ , such that the collection of balls  $\{B_r(z^N)\}_{N \geq 1}$  covers the set  $S_{n,m} \setminus M_\emptyset$ . Without loss of generality, we may assume that there is a positive constant,  $A = A(m, n)$ , such that at most  $A$  balls of the covering have non-empty intersection. Let  $\{\varphi_N\}_{N \geq 0} \subset C_c^\infty(\bar{S}_{n,m})$  be a partition of unity subordinate to the open cover

$$M_\emptyset \cup \bigcup_{N=1}^{\infty} B_r(z^N) = S_{n,m},$$

such that

$$\text{supp } \varphi_0 \subset \{z \in S_{n,m} : \text{dist}(z, \partial S_{n,m}) > r/2\}, \text{ and } \text{supp } \varphi_N \subset \bar{B}_r(z^N), \quad \forall N \geq 1.$$

We may choose the sequence of functions  $\{\varphi_N\}_{N \geq 0}$  such that there is a positive constant,  $c = c(k, m, n)$ , such that

$$\|\varphi_N\|_{C_{WF}^{k,2+\alpha}(\bar{S}_{n,m})} \leq cr^{-(k+3)}, \quad \forall r > 0, \quad \forall N \geq 0. \quad (4.3)$$

We choose a smooth function,  $\psi_0 \in C^\infty(\bar{S}_{n,m})$ , such that  $0 \leq \psi_0 \leq 1$  and

$$\psi_0(z) = \begin{cases} 0, & \text{on } \{z \in S_{n,m} : \text{dist}(z, \partial S_{n,m}) < r/8\}, \\ 1, & \text{on } \{z \in S_{n,m} : \text{dist}(z, \partial S_{n,m}) > r/4\}, \end{cases}$$

and we choose a sequence of functions  $\{\psi_N\}_{N \geq 1}$  such that  $0 \leq \psi_N \leq 1$  and  $B_r(z^N) \subset \{\psi_N = 1\}$ . Thus, we have that

$$\psi_N \varphi_N = \varphi_N, \quad \forall N \geq 0, \quad (4.4)$$

and without loss of generality we may assume that there is a positive constant,  $c = c(k, m, n)$ , with the property that

$$\|\psi_N\|_{C_{WF}^{k,2+\alpha}(\bar{S}_{n,m})} \leq c, \quad \forall r > 0, \quad \forall N \geq 0. \quad (4.5)$$

For  $N = 0$ , let  $L_0$  be a strictly elliptic operator on  $\mathbb{R}^{n+m}$  with  $C^{k,\alpha}([0, T] \times \mathbb{R}^{n+m})$ -Hölder continuous coefficients, such that the operator  $L_0$  agrees with  $L$  on the support of the function  $\psi_0$ . This is possible due to our assumptions (3.1), (3.2) and property (2.3) of the distance function  $\rho$ . We let

$$V_0 : C^{k,\alpha}([0, T] \times \mathbb{R}^{n+m}) \rightarrow \widehat{C}^{k+2,\alpha}([0, T] \times \mathbb{R}^{n+m}),$$

be the solution operator of  $L_0$ , that is, using [17, Theorems 9.2.3 and 8.12.1], we let  $u := V_0 g \in \widehat{C}^{k+2,\alpha}([0, T] \times \mathbb{R}^{n+m})$  be the unique solution to the initial-value problem,  $u_t - L_0 u = g$  on  $(0, T) \times \mathbb{R}^{n+m}$ , and  $u(0, \cdot) = 0$  on  $\mathbb{R}^{n+m}$ , where  $g \in C^{k,\alpha}([0, T] \times \mathbb{R}^{n+m})$ . For each  $N \geq 1$ , there is a set of indices,  $I_N \subseteq \{1, 2, \dots, n\}$ , such that  $B_r(z^N) \subset M'_{I_N}$  and  $\text{supp } \psi_N \subset M''_{I_N}$ . We then

let  $L_N$  be the degenerate-parabolic operator defined by

$$\begin{aligned} L_N u := & \sum_{i \in I_N} x_i a_{ii}(z^N) u_{x_i x_i} + \sum_{i \notin I_N} x_i^N a_{ii}(z^N) u_{x_i x_i} + \sum_{i=1}^n b_i(z^N) u_{x_i} \\ & + \sum_{i,j \in I_N} x_i x_j \tilde{a}_{ij}(z^N) u_{x_i x_j} + \sum_{i \in I_N} \sum_{j \notin I_N} 2x_i x_j^N \tilde{a}_{ij}(z^N) u_{x_i x_j} + \sum_{i,j \notin I_N} x_i^N x_j^N \tilde{a}_{ij}(z^N) u_{x_i x_j} \\ & + \sum_{i \in I_N} \sum_{l=1}^m x_i c_{il}(z^N) u_{x_i y_l} + \sum_{i \notin I_N} \sum_{l=1}^m x_i^N c_{il}(z^N) u_{x_i y_l} + \sum_{l,k=1}^m d_{lk}(z^N) u_{y_l y_k} + \sum_{k=1}^m e_l(z^N) u_{y_k}. \end{aligned}$$

Notice that the differential operator  $L_N$  has the same structure as the operator  $L$  defined in (1.1), with the observation that in this case, the number of ‘degenerate’ coordinates,  $n$ , is replaced by  $n_N := |I_N|$ , and the number of ‘non-degenerate’ coordinates,  $m$ , is replaced by  $m_N := n + m - n_N$ . By relabeling the coordinates, we may assume that the operator  $L_N$  acts on functions defined on a compact manifold with corners,  $P_N$  (see [7, §2.1] for the definition of compact manifolds with corners). We choose the compact manifold  $P_N$  such that

$$\text{supp } \varphi_N \subset B_r(z^N) \subset \{\psi_N \equiv 1\} \subset \text{supp } \psi_N \subset P_N, \quad \forall N \geq 1.$$

We extend the coefficients of the operator  $L_N$  from  $\text{supp } \psi_N$  to  $P_N$  such that it satisfies the hypotheses of [7, Theorem 10.0.2] to conclude that there is a solution operator,

$$V_N : C_{WF}^{k,\alpha}([0, T] \times P_N) \rightarrow \hat{C}_{WF}^{k,2+\alpha}([0, T] \times P_N),$$

such that  $u := V_N g \in \hat{C}_{WF}^{k,2+\alpha}([0, T] \times P_N)$  is the unique solution to the initial-value problem,  $u_t - L_N u = g$  on  $(0, T) \times P_N$ , and  $u(0, \cdot) \equiv 0$  on  $P_N$ , where  $g \in C_{WF}^{k,\alpha}([0, T] \times P_N)$ .

We can now define the operator

$$V : C_{WF}^{k,\alpha}([0, T] \times \bar{S}_{n,m}) \rightarrow \hat{C}_{WF}^{k,2+\alpha}([0, T] \times \bar{S}_{n,m}),$$

by setting

$$Vg := \sum_{N=0}^{\infty} \varphi_N V_N \psi_N g, \quad \forall g \in C_{WF}^{k,\alpha}([0, T] \times \bar{S}_{n,m}).$$

Our goal is to show that inequality (4.2) holds, for small enough values of  $r$  and  $T$ . We have

$$\begin{aligned} (\partial_t - L)Vg - g &= \sum_{N=0}^{\infty} (\partial_t - L)\varphi_N V_N (\psi_N g) - g \\ &= \sum_{N=0}^{\infty} \varphi_N (\partial_t - L)V_N (\psi_N g) - \sum_{N=0}^{\infty} [L, \varphi_N] V_N (\psi_N g) - g, \end{aligned}$$

where the term  $[L, \varphi_N]$  is given by (3.9). Denoting  $u_N := V_N (\psi_N g)$ , for all  $N \geq 0$ , we have

$$\begin{aligned} (\partial_t - L)V_N (\psi_N g) &= -(L - L_N)u_N + (\partial_t - L_N)V_N (\psi_N g) \\ &= -(L - L_N)u_N + \psi_N g, \end{aligned}$$

since  $(\partial_t - L_N)V_N = I$  on  $\text{supp } \psi_N$ , for all  $N \geq 0$ . This implies, by identities (4.4) and the fact that  $\{\varphi_N\}_{N \geq 0}$  is a partition of unity, that

$$(\partial_t - L)Vg - g = - \sum_{N=0}^{\infty} \varphi_N (L - L_N)u_N - \sum_{N=0}^{\infty} [L, \varphi_N] u_N. \quad (4.6)$$

We first estimate the terms in the preceding equality indexed by  $N = 0$ . Because  $L_0 = L$  on the support of  $\psi_0$ , obviously we have that  $\psi_0(L - L_0)u_0 = 0$ . Next, using identity (3.9), there is a positive constant,  $C = C(\alpha, k, K, m, n)$ , such that

$$\begin{aligned} \|[L, \varphi_0] u_0\|_{C^{k,\alpha}([0,T] \times \bar{S}_{n,m})} &\leq C \|\varphi_0\|_{C^{k+2,\alpha}([0,T] \times \bar{S}_{n,m})} \|u_0\|_{C^{k+1,\alpha}([0,T] \times \text{supp } \varphi_0)} \\ &\leq Cr^{-(k+3)} \|u_0\|_{C^{k+1,\alpha}([0,T] \times \text{supp } \varphi_0)} \quad (\text{by (4.3)}). \end{aligned}$$

Using the interpolation inequalities for standard Hölder spaces [17, Theorem 8.8.1], and the fact that the standard parabolic distance and the distance function  $\rho$  are equivalent on  $\text{supp } \varphi_0$ , by (2.3), it follows that there is a positive constant,  $m_k = m_k(\alpha, k, m, n)$ , such that, for all  $\varepsilon \in (0, 1)$ , we have

$$\begin{aligned} \|[L, \varphi_0] u_0\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} &\leq Cr^{-(k+3)} \left( \varepsilon \|u_0\|_{C_{WF}^{k,2+\alpha}([0,T] \times \text{supp } \varphi_0)} \right. \\ &\quad \left. + \varepsilon^{-m_k} \|u_0\|_{C([0,T] \times \text{supp } \varphi_0)} \right). \end{aligned} \quad (4.7)$$

By [17, Theorem 8.12.1] and inequality (4.5), we have that there is a positive constant,  $C = C(\alpha, k, K, m, n, T)$ , such that

$$\|u_0\|_{C_{WF}^{k,2+\alpha}([0,T] \times \text{supp } \varphi_0)} \leq Cr^{-(k+3)} \|g\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})}.$$

From [17, Corollary 8.1.5], it follows that

$$\|u_0\|_{C([0,T] \times \text{supp } \varphi_0)} \leq T \|g\|_{C([0,T] \times \text{supp } \psi_0)},$$

and so, the preceding two inequalities together with (4.7), give us that

$$\|[L, \varphi_0] u_0\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} \leq Cr^{-(k+3)} \left( \varepsilon \|g\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} + \varepsilon^{-m_k} T \|g\|_{C([0,T] \times \bar{S}_{n,m})} \right). \quad (4.8)$$

Next, we estimate the terms in identity (4.6) indexed by  $N \geq 1$ . From identity (4.4), we have that  $\varphi_N(L - L_N)u_N = \varphi_N(L - L_N)(\psi_N u_N)$ . By Lemma 2.7, we have that there are positive constants,  $C_r = C(\alpha, k, K, m, n, r, T)$  and  $m_k = m_k(\alpha, k, m, n)$ , such that for all  $\varepsilon \in (0, 1)$ , we have that

$$\begin{aligned} \|\varphi_N(L - L_N)u_N\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} &= \|\varphi_N(L - L_N)(\psi_N u_N)\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} \\ &\leq \left( \Lambda \|\varphi_N\|_{C(\bar{S}_{n,m})} + C_r \varepsilon \right) \|\psi_N u_N\|_{C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})} \\ &\quad + C_r \varepsilon^{-m_k} \|\psi_N u_N\|_{C([0,T] \times \bar{S}_{n,m})}, \end{aligned}$$

Using the fact that  $0 \leq \psi_N \leq 1$  and  $\text{supp } \psi_N \subset P_N$ , together with inequality (4.5) and [7, Inequality (5.62)], we have that

$$\begin{aligned} \|\varphi_N(L - L_N)u_N\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} &\leq c \left( \Lambda \|\varphi_N\|_{C(\bar{S}_{n,m})} + C_r \varepsilon \right) \|u_N\|_{C_{WF}^{k,2+\alpha}([0,T] \times P_N)} \\ &\quad + C_r \varepsilon^{-m_k} \|u_N\|_{C([0,T] \times P_N)}, \end{aligned}$$

Using the definition of the constant  $\Lambda$  in (2.31), with  $\bar{U}$  replaced by  $\text{supp } \varphi_N$ , from our choice of the operator  $L - L_N$ , and property (3.2) of the coefficients of the operator  $L$ , we obtain that  $\Lambda \leq Cr^{\alpha/2}$ . The preceding estimate becomes

$$\begin{aligned} \|\varphi_N(L - L_N)u_N\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} &\leq C \left( r^{\alpha/2} + C_r \varepsilon \right) \|u_N\|_{C_{WF}^{k,2+\alpha}([0,T] \times P_N)} \\ &\quad + C_r \varepsilon^{-m_k} \|u_N\|_{C([0,T] \times P_N)}. \end{aligned}$$

Applying [7, Proposition 3.3.1] to the function  $\pm u_N(t, z) - t \|g\|_{C([0,T] \times P_N)}$ , we obtain that

$$\|u_N\|_{C([0,T] \times P_N)} \leq T \|g\|_{C([0,T] \times P_N)},$$

while the a priori Schauder estimates [7, Theorem 10.0.2] show that there is a positive constant,  $C = C(\alpha, \delta, k, K, m, n, T)$ , such that

$$\begin{aligned} \|u_N\|_{C_{WF}^{k,2+\alpha}([0,T] \times \text{supp } \varphi_N)} &\leq C \|\psi_N g\|_{C_{WF}^{k,\alpha}([0,T] \times P_N)} \\ &\leq C \|g\|_{C_{WF}^{k,\alpha}([0,T] \times P_N)} \quad (\text{using inequality (4.5)}). \end{aligned}$$

From the preceding three inequalities, it follows that

$$\begin{aligned} \|\varphi_N(L - L_N)u_N\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} &\leq C \left( r^{\alpha/2} + C_r \varepsilon \right) \|g\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} \\ &\quad + C_r \varepsilon^{-m_k} T \|g\|_{C([0,T] \times \bar{S}_{n,m})}. \end{aligned} \quad (4.9)$$

It remains to estimate the term  $[L, \varphi_N]u_N$ , for  $N \geq 1$ , by employing the same method that we used to estimate the term  $[L, \varphi_0]u_0$ . The only change is that we replace the standard interpolation inequalities [17, Theorem 8.8.1] by Corollary 2.4, the standard a priori Schauder estimates [17, Theorems 9.2.3 and 8.12.1] by Theorem 1.2, and the standard maximum principle [17, Corollary 8.1.5] by [7, Proposition 3.3.1]. We then obtain the analogue of inequality (4.8),

$$\begin{aligned} \|[L, \varphi_N]u_N\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} &\leq C r^{-(k+3)} \left( \varepsilon \|g\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} + \varepsilon^{-m_k} T \|g\|_{C([0,T] \times \bar{S}_{n,m})} \right). \end{aligned} \quad (4.10)$$

Combining inequalities (4.8), (4.9) and (4.10), and using the fact that at most  $A$  balls of the covering of  $S_{n,m}$  have non-empty intersection, identity (4.6) yields

$$\begin{aligned} \|(\partial_t - L)Vg - g\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} &\leq C \left( r^{\alpha/2} + \varepsilon r^{-(k+3)} + \varepsilon C_r \right) \|g\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} \\ &\quad + \left( C r^{-(k+3)} \varepsilon^{-m_k} + C_r \right) T \|g\|_{C([0,T] \times \text{supp } \varphi_N)}. \end{aligned}$$

By choosing the positive constants  $r$ ,  $\varepsilon$  and  $T$  small enough, we find a positive constant,  $C_0 < 1$ , such that

$$\|(\partial_t - L)Vg - g\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} \leq C_0 \|g\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})}, \quad \forall g \in C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m}),$$

which is equivalent to (4.2).

The preceding argument implies existence of solutions  $u \in C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})$ , to the inhomogeneous initial-value problem (1.2), with  $f \equiv 0$  and  $g \in C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})$ , up to a fixed time  $T$ . A standard bootstrapping argument allows us to obtain existence of solutions to problem (1.2) up to any time  $T$ .

This completes the proof.  $\square$

Finally, we give the

*Proof of Theorem 1.5.* Uniqueness of solutions is a straightforward consequence of Proposition 4.1, and so, we only consider the question of existence of solutions. Let  $\{f_N\}_{N \geq 1} \subset C^\infty(\bar{S}_{n,m})$  be a sequence of smooth functions such that

$$\|f_N - f\|_{C([0,T] \times \bar{S}_{n,m})} \rightarrow 0, \quad \text{as } N \rightarrow \infty, \quad (4.11)$$

Let  $u_N$  be the unique solution to the inhomogeneous initial-value problem (1.2), with  $u_N(0, \cdot) = f_N$  on  $\bar{S}_{n,m}$ , given by Theorem 1.4. Then, it follows that  $u_N \in C_{WF}^{k,2+\alpha}([0,T] \times \bar{S}_{n,m})$ , for all  $k \in \mathbb{N}$  and all  $\alpha \in (0, 1)$ . From Corollary 4.2 and property (4.11), we obtain that

$$\|u_N - u_M\|_{C([0,T] \times \bar{S}_{n,m})} \leq \|f_N - f_M\|_{C([0,T] \times \bar{S}_{n,m})} \rightarrow 0, \quad \text{as } N, M \rightarrow \infty,$$



and so, the sequence  $\{u_N\}_{N \geq 1}$  converges uniformly to a function  $u \in C([0, T] \times \bar{S}_{n,m})$ , and clearly we have that  $u(0, \cdot) = f$  on  $\bar{S}_{n,m}$ . Moreover, applying Corollary 4.2 to each of the functions  $u_N$ , and using property (4.11), we have that

$$\|u\|_{C([0,T] \times \bar{S}_{n,m})} \leq C \|f\|_{C([0,T] \times \bar{S}_{n,m})}. \quad (4.12)$$

Let  $k \in \mathbb{N}$ ,  $\alpha \in (0, 1)$ ,  $T_0 > 0$  and let  $r_0 = r_0(\alpha, k, m, n)$  be the positive constant appearing in the conclusion of Theorem 1.1. Covering  $\bar{S}_{n,m}$  by a countable collection of balls  $\{B_{r_0}(z^N)\}_{N \geq 1}$ , we may apply estimate (1.3) on each ball, to obtain that there is a positive constant,  $C = C(\alpha, \delta, k, K, m, n, T_0, T)$ , such that

$$\|u_N\|_{C_{WF}^{k,2+\alpha}([T_0,T] \times \bar{S}_{n,m})} \leq C \left( \|\partial_t u_N - Lu_N\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} + \|u_N\|_{C([0,T] \times \bar{S}_{n,m})} \right),$$

and using inequality (4.12), we have that

$$\|u_N\|_{C_{WF}^{k,2+\alpha}([T_0,T] \times \bar{S}_{n,m})} \leq C \left( \|g\|_{C_{WF}^{k,\alpha}([0,T] \times \bar{S}_{n,m})} + \|f\|_{C([0,T] \times \bar{S}_{n,m})} \right), \quad \forall N \geq 1.$$

Thus, applying the Arzelà-Ascoli Theorem and [7, Proposition 5.2.8], we can find a subsequence of  $\{u_N\}_{N \geq 1}$ , which converges uniformly on compact subsets of  $[T_0, T] \times \bar{S}_{n,m}$  in the Hölder space  $C_{WF}^{k,2+\alpha'}([T_0, T] \times \bar{S}_{n,m})$ , for all  $\alpha' \in (0, \alpha)$ , to a function that belongs to  $C_{WF}^{k,2+\alpha}([T_0, T] \times \bar{S}_{n,m})$ . Thus, the limit function,  $u \in C([0, T] \times \bar{S}_{n,m})$ , belongs to  $C_{WF}^{k,2+\alpha}([T_0, T] \times \bar{S}_{n,m})$ , satisfies the Schauder estimate (1.6), for all  $k \in \mathbb{N}$  and  $\alpha \in (0, 1)$ , and solves the inhomogeneous initial-value problem (1.2). This completes the proof.  $\square$

## REFERENCES

- [1] S. R. Athreya, M. T. Barlow, R. F. Bass, and E. A. Perkins, *Degenerate stochastic differential equations and super-Markov chains*, Probab. Theory Related Fields **123** (2002), 484–520.
- [2] R. F. Bass and E. A. Perkins, *Degenerate stochastic differential equations with Hölder continuous coefficients and super-Markov chains*, Trans. Amer. Math. Soc. **355** (2003), 373–405.
- [3] P. Daskalopoulos and R. Hamilton,  *$C^\infty$ -regularity of the free boundary for the porous medium equation*, J. Amer. Math. Soc. **11** (1998), 899–965.
- [4] C. L. Epstein and R. Mazzeo,  *$C^0$ -estimates for degenerate diffusion operators arising in population biology*, pp. 65, preprint.
- [5] ———,  *$C^0$ -estimates for diagonal degenerate diffusion operators arising in population biology*, pp. 19, preprint.
- [6] ———, *Wright-Fisher diffusion in one dimension*, SIAM J. Math. Anal. **42** (2010), 568–608.
- [7] ———, *Degenerate diffusion operators arising in population biology*, Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 2013, arXiv:1110.0032.
- [8] C. L. Epstein and C. A. Pop, *Harnack inequalities for degenerate diffusions*, pp. 55, preprint.
- [9] S. N. Ethier and T. G. Kurtz, *Markov processes: Characterization and convergence*, Wiley, 1985.
- [10] P. M. N. Feehan and C. A. Pop, *A Schauder approach to degenerate-parabolic partial differential equations with unbounded coefficients*, Journal of Differential Equations **254** (2013), 4401–4445, arXiv:1112.4824.
- [11] R. A. Fisher, *On the dominance ratio*, Proc. Roy. Soc. Edin. **42** (1922), 321–431.
- [12] J. B. S. Haldane, *The causes of evolution*, Harper and Brothers, New York, 1932.
- [13] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, second ed., Springer, New York, 1991.
- [14] S. Karlin and Taylor, *A second course on stochastic processes*, Academic, New York, 1981.
- [15] M. Kimura, *Some problems of stochastic processes in genetics*, Ann. Math. Statist. **28** (1957), 882–901.
- [16] ———, *Diffusion models in population genetics*, J. Appl. Probability **1** (1964), 177–232.
- [17] N. V. Krylov, *Lectures on elliptic and parabolic equations in Hölder spaces*, American Mathematical Society, Providence, RI, 1996.
- [18] J. Moser, *A Harnack inequality for parabolic differential equations*, Comm. Pure Appl. Math. **17** (1964), 101–134.
- [19] ———, *Correction to: “A Harnack inequality for parabolic differential equations”*, Comm. Pure Appl. Math. **20** (1967), 231–236.

- [20] ———, *On a pointwise estimate for parabolic differential equations*, Comm. Pure Appl. Math. **24** (1971), 727–740.
- [21] C. A. Pop, *Existence, uniqueness and the strong Markov property of solutions to Kimura stochastic differential equations with singular drift*, pp. 25, preprint.
- [22] N. Shimakura, *Formulas for diffusion approximations of some gene frequency models*, J. Math. Kyoto Univ. **21** (1981), no. 1, 19–45.
- [23] S. Wright, *Evolution in Mendelian populations*, Genetics **16** (1931), 97–159.

(CP) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, 209 SOUTH 33RD STREET, PHILADELPHIA, PA 19104-6395

*E-mail address:* `cpop@math.upenn.edu`